Global Aspects of Ergodic Group Actions

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Contents

Introduction

(A) The original motivation for this work came from the study of certain results in ergodic theory, primarily, but not exclusively, obtained over the last several years. These included: i) Work of Hjorth and Foreman-Weiss concerning the complexity of the problem of classification of ergodic measure preserving transformations up to conjugacy, ii) Various results concerning the structure of the outer automorphism group of a countable measure preserving equivalence relation, including, e.g., work of Jones-Schmidt, iii) Ergodic theoretic characterizations of groups with property (T) or the Haagerup Approximation Property, in particular results of Schmidt, Connes-Weiss, Jolissaint, and Glasner-Weiss, iv) Results of Hjorth, Popa, Gaboriau-Popa and Törnquist on the existence of many non-orbit equivalent ergodic actions of certain non-amenable groups, v) Popa's recent work on cocycle superrigidity.

Despite the apparent diversity of the subjects treated in these works, we gradually realized that they can be understood within a rather unified framework. This is the study of the global structure of the space $A(\Gamma, X, \mu)$ of measure preserving actions of a countable group Γ on a standard measure space (X, μ) and the canonical action of the automorphism group $\mathrm{Aut}(X, \mu)$ of (X, μ) by conjugation on $A(\Gamma, X, \mu)$ as well as the study of the global structure of the space of cocycles and certain canonical actions on it. Our goal here is to explore this point of view by presenting (a) earlier results, sometimes in new formulations or with new proofs, (b) new theorems, and finally (c) interesting open questions that are suggested by this approach.

(B) The book is divided into three chapters, the first consisting of Sections 1–9, the second of Sections 10–18, and the third of Sections 19–30. There are also nine appendices.

In the first chapter, we study the automorphism group $\mathrm{Aut}(X, \mu)$ of a standard measure space (i.e., the group of measure preserving automorphisms of (X, μ)) and various subgroups associated with measure preserving equivalence relations. Note that $\mathrm{Aut}(X, \mu)$ can be also identified with the space $A(\mathbb{Z}, X, \mu)$ of measure preserving $\mathbb{Z}$-actions. Sections 1, 2 review some basic facts about the group $\mathrm{Aut}(X, \mu)$. In Section 2 we also show that the class of mild mixing transformations in $\mathrm{Aut}(X, \mu)$ is co-analytic but not Borel (a result also proved independently by Robert Kaufman). This is in contrast with the well-known fact that the ergodic, weak mixing and (strong) mixing transformations are, resp., G_δ, G_δ and $F_{\sigma\delta}$ sets. In Section

3 we discuss the full group $[E] \subseteq \mathrm{Aut}(X, \mu)$ of a measure preserving countable Borel equivalence relation E on (X, μ). In Section 4 we give a detailed proof of Dye's reconstruction theorem, which asserts that the equivalence relation E is determined up to (measure preserving) isomorphism by $[E]$ *as an abstract group*. In Section 5 we give a new method for proving the turbulence property of the conjugacy action of $\mathrm{Aut}(X, \mu)$ on the set ERG of ergodic transformations in $\mathrm{Aut}(X, \mu)$, originally established by Foreman-Weiss, and use this method to prove other turbulence results in the context of full groups. We also extend the work of Hjorth and Foreman-Weiss on non-classification by countable structures of weak mixing transformations in $\mathrm{Aut}(X, \mu)$, up to conjugacy or unitary (spectral) equivalence, to the case of mixing transformations (and obtain more precise information in this case). In Sections 6, 7 we review the basic properties of the automorphism group $N[E]$ of a measure preserving countable Borel equivalence relation E on (X, μ) and its outer automorphism group, $\mathrm{Out}(E)$, and establish (in Section 7) the turbulence of the latter, when E is hyperfinite (and even in more general situations). Understanding when $\mathrm{Out}(E)$ is a Polish group (with respect to the canonical topology discussed in Section 7) is an interesting open problem raised in work of Jones-Schmidt. In Section 8 we establish a connection between the Polishness of $\mathrm{Out}(E)$ and Gaboriau's theory of costs and use it to obtain a partial answer to this problem. In Section 9 we discuss Effros' notion of inner amenability and its relationship with the open problem of Schmidt of whether this property of a group is characterized by the failure of Polishness of the outer automorphism group of some equivalence relation induced by a free, measure preserving, ergodic action of the group. Some known and new results related to this concept and Schmidt's problem are presented here.

In the second chapter, we start (in Section 10) with establishing some basic properties of the space $A(\Gamma, X, \mu)$ of measure preserving actions of a countable group Γ on (X, μ). For finitely generated groups, we also calculate an upper bound for the descriptive complexity of the cost function on this space and use this to show that the generic action realizes the cost of the group. In Section 11 we recall several known ergodic theoretic characterizations of groups with property (T) or the Haagerup Approximation Property (HAP). In Section 12 we study the structure of the space of ergodic actions (and some of its subspaces) in $A(\Gamma, X, \mu)$, recasting in this context characterizations of Glasner-Weiss and Glasner concerning property (T) or the HAP, originally formulated in terms of the structure of extreme points in the space of invariant measures. In Section 13 we study, using the method introduced in Section 5, turbulence of conjugacy in the space of actions and also discuss work of Hjorth and Foreman-Weiss concerning non-classification by countable structures of such actions, up to conjugacy. We also prove an analogous result for unitary (spectral) equivalence. In Section 14, we present the essence of Hjorth's result on the existence of many non-orbit equivalent actions of property (T) groups as a basic property concerning the topological structure of the conjugacy classes of ergodic actions of such groups. We use Hjorth's method to show that for such groups the set of ergodic actions is clopen in the uniform topology and so is each conjugacy class of ergodic actions. In Section 15 we study connectedness properties in the space of actions, using again the method of Section 5. This illustrates the close connection between local connectedness properties and turbulence. We show, in particular, that the space $A(\Gamma, X, \mu)$ is path-connected in the weak topology. We also contrast this to the work in Section 14 to point out the interesting phenomenon that connectedness properties in the space of actions of a group seem to be related to properties of the group, such as amenability or property (T). In particular, for groups Γ with property (T), we determine completely the path components of the space $A(\Gamma, X, \mu)$ in the uniform topology. In Section 16 we discuss results of Popa concerning the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$. These are used in Section 17, along with other ideas, in the proof of a non-classification result of Törnquist for orbit equivalence of actions of non-abelian free groups. We also briefly discuss very recent results of Ioana, Epstein and Epstein-Ioana-Kechris-Tsankov that extend this to arbitrary non-amenable groups. Finally, Section 18 contains a survey of classification problems concerning group actions.

In the third chapter, we give in Section 19 a short introduction to the properties of the group of group-valued random variables and then in Sections 20, 21 we discuss the space of cocycles of a group action or an equivalence relation and some of the invariants associated with such cocycles, like the associated Mackey action and the essential range. We also discuss cocycles arising from reductions and homomorphisms of equivalence relations. Our primary interest is in cocycles with countable (discrete) targets. The next two Sections 22 and 23 contain background material concerning continuous and isometric group actions and Effros' Theorem. The topology of the space of cocycles is discussed in Section 24 and the study of the global properties of the cohomology equivalence relation is the subject of the final Sections 25–30. Section 25 contains some general properties of the cohomology relation, and Section 26 is concerned with the hyperfinite case. There is a large literature here but it is not our main focus in this work. There is a fundamental dichotomy in the structure of the cohomology relation for the cocycles of a given equivalence relation (or action), which in some form is already present in the work of Schmidt for the case of cocycles with abelian targets. In a precise sense, that is explained in these sections, when an equivalence relation E is *non* E_0*-ergodic* (or *not strongly ergodic*), the structure of the cohomology relation on its cocycles is very complicated. This is the subject in Section 27. On the other hand, when the equivalence relation E is E_0*-ergodic* (or *strongly ergodic*), then the cohomology relation is simple, i.e., smooth, provided the target groups satisfy the so-called *minimal condition on centralizers*, discussed in Section 28 (and these include, e.g., the abelian and the linear groups). If the target groups do not satisfy this condition, in certain situations, like, e.g., when E is given by an action of the free group with infinitely many generators, the cohomology relation is not smooth. Thus in the E_0-ergodic case, there is an additional dichotomy having to do with the structure of the target groups. This is the topic of Section 29. Finally in Section 30 we deal with the special case of actions of groups with property (T) and discuss some recent results of Popa on cocycle superrigidity. We also establish in the above sections some new characterizations of amenable and property (T) groups, that, in particular, extend earlier results of Schmidt and also show that there is a positive link between the E_0-ergodicity of an equivalence relation E and the Polishness of its outer automorphism group, $\text{Out}(E)$, an issue raised by Jones-Schmidt. They have pointed out that E_0-ergodicity does not in general imply that $\text{Out}(E)$ is Polish but we show in Section 29 that one has a positive implication when E is induced by a free action of a group with the minimal condition on centralizers.

Appendices A – I present background material concerning Hilbert spaces and tensor products, Gaussian probability spaces and the Wiener chaos decomposition, several relevant aspects of the theory of unitary representations (including unitary representations of abelian groups and induced representations as well as some basic results about the space of unitary representations of a group) and finally semidirect products of groups. We also include, in Appendix E, a detailed proof of the standard result that any unitary representation of a countable group is a subrepresentation of the Koopman representation associated with some measure preserving action of that group.

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CHAPTER I

Measure preserving automorphisms

1. The group Aut(X, μ)

(A) By a *standard measure space* (X, μ) we mean a standard Borel space X with a non-atomic probability Borel measure μ. All such spaces are isomorphic to $([0,1], \lambda)$, where λ is Lebesgue measure on the Borel subsets of [0,1]. Denote by MALG_μ the *measure algebra* of μ, i.e., the algebra of Borel subsets of X, modulo null sets. It is a Polish Boolean algebra under the topology given by the complete metric

$$d(A, B) = d_\mu(A, B) = \mu(A \Delta B),$$

where Δ is *symmetric difference.*

Throughout we write $L^2(X, \mu) = L^2(X, \mu, \mathbb{C})$ for the Hilbert space of complex-valued square-integrable functions. When we want to refer explicitly to the space of real-valued functions, we will write it as $L^2(X, \mu, \mathbb{R})$.

We denote by $\mathrm{Aut}(X, \mu)$ the group of Borel automorphisms of X which preserve the measure μ and in which we identify two such automorphisms if they agree μ-a.e.

Convention. *In the sequel, we will usually ignore null sets, unless there is a danger of confusion.*

There are two fundamental group topologies on $\mathrm{Aut}(X, \mu)$: the *weak* and the *uniform* topology, which we now proceed to describe.

(B) The *weak topology* on $\mathrm{Aut}(X, \mu)$ is generated by the functions

$$T \mapsto T(A), \quad A \in \mathrm{MALG}_\mu,$$

(i.e., it is the smallest topology in which these maps are continuous). With this topology, denoted by w, $(\mathrm{Aut}(X, \mu), w)$ is a Polish topological group. A left-invariant compatible metric is given by

$$\delta_w(S, T) = \sum 2^{-n} \mu(S(A_n) \Delta T(A_n)),$$

where $\{A_n\}$ is a dense set in MALG_μ (e.g., an algebra generating the Borel sets of X), and a complete compatible metric by

$$\bar{\delta}_w(S, T) = \delta_w(S, T) + \delta_w(S^{-1}, T^{-1}).$$

We can also view $\mathrm{Aut}(X, \mu)$ as the automorphism group of the measure algebra (MALG_μ, μ) (equipped with the pointwise convergence topology), as well as the closed subgroup of the isometry group $\mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$ (again equipped with the pointwise convergence topology) consisting of all $T \in \mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$ with $T(\emptyset) = \emptyset$. (We only need to verify that an isometry T with $T(\emptyset) = \emptyset$ is an automorphism of the Boolean algebra MALG_μ. First notice that $\mu(A) = d_\mu(\emptyset, A) = d_\mu(\emptyset, T(A)) = \mu(T(A))$. Next we see that if $A, B \in \mathrm{MALG}_\mu$ and $A \cap B = \emptyset$, then $T(A) \cap T(B) = \emptyset$. Indeed $d_\mu(A, B) = \mu(A) + \mu(B)$, thus $d_\mu(T(A), T(B)) = d_\mu(A, B) = \mu(A) + \mu(B) = \mu(T(A)) + \mu(T(B))$, so $T(A) \cap T(B) = \emptyset$. From this we obtain that $T(\sim A) = \sim T(A)$ (where $\sim A = X \setminus A$), since $d_\mu(A, \sim A) = 1$, so $d_\mu(T(A), T(\sim A)) = 1$ and $T(A) \cap T(\sim A)) = \emptyset$, so $T(\sim A) = \sim T(A)$. Finally, $A \subseteq B \Leftrightarrow A \cap (\sim B) = \emptyset \Leftrightarrow T(A) \cap (\sim T(B)) = \emptyset \Leftrightarrow T(A) \subseteq T(B)$, so T is an automorphism of the Boolean algebra.)

Identifying $T \in \mathrm{Aut}(X, \mu)$ with the unitary operator on $L^2(X, \mu)$ defined by

$$U_T(f) = f \circ T^{-1},$$

we can identify $\mathrm{Aut}(X, \mu)$ (as a topological group) with a closed subgroup of $U(L^2(X, \mu))$, the unitary group of the Hilbert space $L^2(X, \mu)$, equipped with the weak topology, which coincides on the unitary group with the strong topology. With this identification, $\mathrm{Aut}(X, \mu)$ becomes the group of $U \in U(L^2(X, \mu))$ that satisfy:

$$U(fg) = U(f)U(g) \text{ (pointwise multiplication)}$$

whenever $f, g, fg \in L^2(X, \mu)$, i.e., the so-called *multiplicative operators*. It also coincides with the group of $U \in U(L^2(X, \mu))$ that satisfy

$$f \geq 0 \Rightarrow U(f) \geq 0,$$

$$U(1) = 1,$$

i.e., the so-called *positivity preserving operators* fixing 1. Note also that any such U preserves real functions.

Let $L^2_0(X, \mu) = \mathbb{C}^\perp = \{f \in L^2(X, \mu) : \int f d\mu = 0\}$ be the orthogonal of the space $\mathbb{C} = \mathbb{C}1$ of the constant functions. Clearly U_T is the identity on $\mathbb{C}$ and $L^2_0(X, \mu)$ is invariant under U_T. So we can also identify T with $U^0_T = U_T | L^2_0(X, \mu)$ and view $\mathrm{Aut}(X, \mu)$ as a closed subgroup of $U(L^2_0(X, \mu))$.

The map $T \mapsto U_T$ of $\mathrm{Aut}(X, \mu)$ into $U(L^2(X, \mu))$ is called the *Koopman representation* of $\mathrm{Aut}(X, \mu)$. It has no non-trivial closed invariant subspaces in $L^2_0(X, \mu)$, i.e., the Koopman representation on $L^2_0(X, \mu)$ is irreducible (see Glasner, [Gl2], 5.14).

Finally, for further reference, let us describe explicitly some open bases for $(\mathrm{Aut}(X, \mu), w)$. The sets of the form

$$V_{S, A_1 \dots A_n, \epsilon} = \{T : \forall i \leq n(\mu(T(A_i) \Delta S(A_i)) < \epsilon)\},$$

where $A_1, \dots, A_n \in \mathrm{MALG}_\mu, \epsilon > 0, S \in \mathrm{Aut}(X, \mu)$, form an open basis for w. This is immediate from the definition of this topology.

Next we claim that the sets of the form

$$W_{S, A_1 \dots A_n, \epsilon} = \{T : \forall i, j \leq n(|\mu(S(A_i) \cap A_j) - \mu(T(A_i) \cap A_j)| < \epsilon)\},$$

for $A_1,\dots,A_n \in \mathrm{MALG}_\mu, \epsilon > 0, S \in \mathrm{Aut}(X,\mu)$, form an open basis for w. The easiest way to see this, as pointed out by O. Ageev, is to use the identification of $(\mathrm{Aut}(X,\mu),w)$ with a closed subgroup of $U(L^2(X,\mu))$ with the weak topology, which is generated by the functions $U \mapsto \langle U(f),g\rangle = \int U(f)\overline{g}d\mu$, for $f,g \in L^2(X,\mu)$. Simply notice that if we denote by χ_A the *characteristic function* of a set A, then the linear combinations $\sum_{i=1}^n \alpha_i\chi_{A_i}$ of characteristic functions of Borel sets form a dense set in the Hilbert space $L^2(X,\mu)$ and $\langle U_T(\chi_A),\chi_B\rangle = \int(\chi_A \circ T^{-1})\chi_B d\mu = \mu(T(A)\cap B)$.

It is clear that we can restrict in $V_{S,A_1\dots A_{n,\epsilon}}$ or $W_{S,A_1\dots A_{n,\epsilon}}$ the sets $A_1,\dots,A_n$ to any countable dense set in MALG_μ and ϵ to rationals and still obtain bases. Also it is easy to check that we can restrict above the $A_1,\dots,A_n$ to belong to any countable dense (Boolean) subalgebra of MALG_μ and moreover assume that $A_1,\dots,A_n$ form a partition of X (by looking, for each $A_1,\dots,A_n$, at the atoms of the Boolean algebra generated by $A_1,\dots,A_n$).

(C) We now come to the *uniform topology*, u. This is defined by the metric

$$\delta'_u(S,T) = \sup_{A\in\mathrm{MALG}_\mu} \mu(S(A)\Delta T(A)).$$

This is a 2-sided invariant complete metric on $\mathrm{Aut}(X,\mu)$ but it is not separable (e.g., if S_α denotes the rotation by $\alpha \in \mathbb{T}$ of the group $\mathbb{T}$, then $\{S_\alpha\}$ is discrete). Clearly $w \subsetneq u$.

An equivalent to δ'_u metric is defined by

$$\delta_u(S,T) = \mu(\{x : S(x) \neq T(x)\}).$$

In fact,

$$\frac{2}{3}\delta_u \leq \delta'_u \leq \delta_u.$$

The metric δ_u is also 2-sided invariant. For $T : X \to X$ let

$$\mathrm{supp}(T) = \{x : T(x) \neq x\}.$$

Then $\delta_u(T,1) = \mu(\mathrm{supp}(T))$, where $1 = \mathrm{id}$ (the identity function on X).

We note that each closed δ'_u-ball is closed in w and therefore each open δ'_u-ball is F_σ in w. To see this, note that for each $T_0 \in \mathrm{Aut}(X,\mu), \epsilon > 0$, the ball

$$\{T : \delta'_u(T_0,T) \leq \epsilon\}$$

coincides with the set of T satisfying

$$\forall A \in \mathrm{MALG}_\mu(d_\mu(T(A),T_0(A)) \leq \epsilon).$$

Similarly, each closed δ_u-ball is closed in w. To see this, it is enough to show, for each $\epsilon > 0$, that the set $\{T : \delta_u(T,1) > \epsilon\}$ is open in w. Fix T with $\delta_u(T,1) > \epsilon$. Then we can find pairwise disjoint Borel sets $A_1,\dots,A_n$ with $\sum_{i=1}^n \mu(A_i) > \epsilon$ and $T(A_i)\cap A_i = \emptyset, \forall i \leq n$. Fix $\delta < \frac{1}{n}(\sum_{i=1}^n \mu(A_i) - \epsilon)$. Then if $\mu(S(A_i)\Delta T(A_i)) < \delta, \forall i \leq n$, we have

$$\mu(S(A_i)\Delta A_i) \geq \mu(T(A_i)\Delta A_i) - \mu(S(A_i)\Delta T(A_i)) > 2\mu(A_i) - \delta,$$

so $\mu(A_i \setminus S(A_i)) > \mu(A_i) - \delta, \forall i \leq n$. Then $\delta_u(S,1) \geq \sum_{i=1}^n \mu(A_i \setminus S(A_i)) > \sum_{i=1}^n \mu(A_i) - n\delta > \epsilon$.

It follows that both δ_u and δ'_u are lower-semicontinuous in the space $(\mathrm{Aut}(X,\mu), w)^2$.

Let us finally observe that similar calculations show that the map

$$T \mapsto \mathrm{supp}(T)$$
$$\mathrm{Aut}(X,\mu) \to \mathrm{MALG}_\mu$$

is continuous in $(\mathrm{Aut}(X,\mu), u)$ and Baire class 1 in $(\mathrm{Aut}(X,\mu), w)$. This is clear for u. For the weak topology, we need to show that for each $A \in \mathrm{MALG}_\mu, \epsilon > 0$, the set $\{T : \mu(\mathrm{supp}(T)\Delta A) < \epsilon\} = F$ is F_σ. Note that

$$\begin{aligned} T \in F \;\Leftrightarrow\; & \exists \alpha, \beta[\alpha, \beta \in \mathbb{Q} \text{ and } \alpha, \beta \geq 0 \text{ and } \alpha + \beta < \epsilon \text{ and} \\ & \mu(\{x \notin A : T(x) \neq x\}) \leq \alpha \text{ and} \\ & \mu(\{x \in A : T(x) = x\}) < \beta]. \end{aligned}$$

Now the previous argument also shows that for each $\gamma > 0, B \in \mathrm{MALG}_\mu$,

$$\{T : \mu(\{x \in B : T(x) \neq x\}) > \gamma\}$$

is open in w. It follows that

$$\{T : \mu(\{x \notin A : T(x) \neq x\}) \leq \alpha\}$$

is closed in w and since $\mu(\{x \in A : T(x) = x\}) = \mu(A) - \mu(\{x \in A : T(x) \neq x\})$, the set

$$\{T : \mu(\{x \in A : T(x) = x\}) < \beta\}$$

is open in w, thus F is F_σ in w. □

Remark. One could also consider the operator norm topology on the group $\mathrm{Aut}(X,\mu)$ induced by the identification $T \mapsto U_T$. However, it is easy to see that this is the discrete topology. We will show that $T \neq 1 \Rightarrow \|U_T - I\| \geq 1$ (where I is the identity operator on $L^2(X,\mu)$). Since $T \neq 1$ there is a set of positive measure on which $T(x) \neq x$, so there is a set A of positive measure, say ϵ, such that $T(A) \cap A = \emptyset$. Let $f \in L^2(X,\mu)$ be equal to $\frac{1}{\sqrt{\epsilon}}$ on A and 0 otherwise. Thus $\|f\| = 1$. Also $\|U_T(f) - f\|^2 = \int |f(T^{-1}(x)) - f(x)|^2 d\mu(x) \geq \int_{x \in A} |f(T^{-1}(x)) - f(x)|^2 d\mu(x) = 1$.

Convention. *When we consider* $\mathrm{Aut}(X,\mu)$ *as a topological group without explicitly indicating which topology we use, it will be assumed that it is equipped with the weak topology.*

Comments. The basic facts about the topologies w, u can be found in Halmos [Ha]. Note however that Halmos uses δ'_u for our δ_u (and vice versa). Since we more often use (our) δ_u we decided to leave it unprimed. For the characterizations of the unitary operators $U_T, T \in \mathrm{Aut}(X,\mu)$, see, e.g., Fleming-Jamison [FJ], p. 73, Glasner [Gl2], p. 366, and Walters [Wa], Ch. 2.

2. Some basic facts about $\mathrm{Aut}(X,\mu)$

(A) An element $T \in \mathrm{Aut}(X,\mu)$ is *periodic* if all its orbits are finite a.e. It is *periodic* of period n if for almost all $x, T^0(x) = x, T^1(x), \ldots, T^{n-1}(x)$ are distinct but $T^n(x) = x$. It is *aperiodic* if $T^n(x) \neq x$ for all $n \neq 0$, a.e.

Consider now $X = 2^{\mathbb{N}}, \mu =$ the usual product measure. A *basic nbhd* of rank n is a set of the form

$$N_s = \{x \in 2^{\mathbb{N}} : s \subseteq x\},$$

where $s \in 2^n$. A *dyadic permutation* of rank n is determined by a permutation π of 2^n via

$$T_\pi(s\hat{\ }x) = \pi(s)\hat{\ }x.$$

It is clearly in $\mathrm{Aut}(X,\mu)$. If π is cyclic, we call the corresponding T_π a *cyclic dyadic permutation* of rank n. In that case T_π is periodic of period 2^n.

We now have the following approximation theorem.

Theorem 2.1 (Weak Approximation Theorem, Halmos [Ha]). *The cyclic dyadic permutations are dense in* $(\mathrm{Aut}(X,\mu), w)$. *In fact, every open set contains cyclic dyadic permutations of any sufficiently high rank.*

Remark. The following nice argument for the special case of 2.1 that states that a measure preserving homeomorphism of X can be weakly approximated by a dyadic permutation comes from [LS], where it is attributed to Steger.

Fix $T \in \mathrm{Aut}(X,\mu), T$ a homeomorphism. It is enough to show that for any given $\epsilon > 0$ and $f_1, \ldots, f_m \in C(X)$, there is n such that for any $N \geq n$, there is π of rank N such that $|f_\ell(T_\pi^{-1}(x)) - f_\ell(T^{-1}(x))| < \epsilon, \forall x \in X, \ell = 1, \ldots, m$. Choose n large enough so that the oscillation of every $f_\ell, f_\ell \circ T^{-1}$ on each basic nbhd of rank n is $< \epsilon/2$. Fix now $N \geq n$ and enumerate the basic nbhds of rank N as $D_1, \ldots, D_{2^N}$. Next define a bipartite graph G on two disjoint copies of $\{1, \ldots, 2^N\}$ by:

$$(i,j) \in G \Leftrightarrow D_i \cap T(D_j) \neq \emptyset.$$

Claim. *This satisfies the criterion for the application of the marriage theorem.*

Granting this, by the marriage theorem, there is a bijection

$$\varphi : \{1, \ldots, 2^N\} \to \{1, \ldots, 2^N\}$$

with $D_i \cap T(D_{\varphi(i)}) \neq \emptyset$. Consider now $x \in X$. Say $x \in D_i$, so that $T_\varphi(x) \in D_{\varphi(i)}$. Also $T^{-1}(D_i) \cap D_{\varphi(i)} \neq \emptyset$, so find $z \in D_i$ with $T^{-1}(z) = y \in D_{\varphi(i)}$ Then $|f_\ell(T_\varphi(x)) - f_\ell(y)| < \epsilon/2$ and $|f_\ell(y) - f_\ell(T^{-1}(x))| < \epsilon/2$, thus $|f_\ell(T_\varphi(x)) - f_\ell(T^{-1}(x))| < \epsilon$. Finally, put $\pi = \varphi^{-1}$.

Proof of the claim. Fix $A \subseteq \{1, \ldots, 2^N\}$ of cardinality M. We will check that there are $\geq M$ points adjacent to elements of A. If not, then there is $B \subseteq \{1, \ldots, 2^N\}$ of cardinality $> 2^N - M$ such that $D_i \cap T(D_j) = \emptyset$ for all $i \in A, j \in B$. Then $(\bigcup_{i \in A} D_i) \cap (\bigcup_{j \in B} T(D_j)) = \emptyset$, so $1 \geq \sum_{i \in A} \mu(D_i) +$

$\sum_{j\in B}\mu(T(D_j)) = \frac{M}{2^N} + \sum_{j\in B}\mu(D_j) > \frac{M}{2^N} + \frac{2^N-M}{2^N} = 1$, a contradiction. Similarly reversing the roles of i, j. $\square$

In particular, 2.1 implies that $(\mathrm{Aut}(X,\mu), w)$ is *topologically locally finite*, i.e., has a locally finite countable dense subgroup.

We also have the following result which is a consequence of the Rokhlin Lemma.

Theorem 2.2 (Uniform Approximation Theorem, Rokhlin, Halmos [Ha]). *If $T \in \mathrm{Aut}(X,\mu)$ is aperiodic, then for each $N \geq 1, \epsilon > 0$ there is a periodic $S \in \mathrm{Aut}(X,\mu)$ of period N such that $\delta_u(S,T) \leq \frac{1}{N} + \epsilon$.*

Consider now the set APER of all aperiodic elements of $\mathrm{Aut}(X,\mu)$. Then as

$$T \in \mathrm{APER} \Leftrightarrow \forall n\ (\delta_u(T^n, 1) = 1) \Leftrightarrow \forall n \forall m \left(\delta_u(T^n,1) > 1 - \frac{1}{m}\right),$$

APER is G_δ in $(\mathrm{Aut}(X,\mu), w)$. (It is also clearly closed in $(\mathrm{Aut}(X,\mu), u)$.) In fact the following holds.

Proposition 2.3. APER *is dense G_δ in* $(\mathrm{Aut}(X,\mu), w)$.

Proof. Take $X = 2^{\mathbb{N}}$ with the usual measure μ. Then the sets of the form

$$\bigcap_{i=1}^{k}\{T : d_\mu(T(D_i), T_0(D_i)) < \epsilon\},$$

D_i finite unions of basic nbhds, $T_0 \in \mathrm{Aut}(X,\mu), \epsilon > 0$ form a nbhd basis for T_0 in w. Thus it is enough for each dyadic permutation π, of rank say n, to find $T \in \mathrm{APER}$ such that for each $s \in 2^n, T(N_s) = T_\pi(N_s) = N_{\pi(s)}$. Let $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be any aperiodic element of $\mathrm{Aut}(X,\mu)$, e.g., the odometer: $\varphi(1^n\hat{\ }0\hat{\ }x) = 0^n\hat{\ }1\hat{\ }x, \varphi(1^\infty) = 0^\infty$. Define then $T(s\hat{\ }x) = \pi(s)\hat{\ }\varphi(x)$. Clearly this works. $\square$

Moreover, using the Uniform Approximation Theorem, one obtains the following result.

Theorem 2.4 (Conjugacy Lemma, Halmos [Ha]). *Let $T \in$ APER. Then its conjugacy class $\{STS^{-1} : S \in \mathrm{Aut}(X,\mu)\}$ is uniformly dense in* APER. *Therefore its conjugacy class is weakly dense in* $\mathrm{Aut}(X,\mu)$. *Conversely, if the conjugacy class of T is weakly dense, $T \in$ APER.*

Proof. Observe that if U, V are periodic of period N, then they are conjugate. Indeed let $A \subseteq X$ be a Borel set that meets (almost) every orbit of U in exactly one point and let $B \subseteq X$ be defined similarly for V. Then $\mu(A) = \mu(B) = \frac{1}{N}$. Thus there is $S_0 \in \mathrm{Aut}(X,\mu)$ with $S_0(A) = B$. Define now $S \in \mathrm{Aut}(X,\mu)$ by

$$S(x) = S_0(x),\ \text{if } x \in A,$$
$$S(U^n(x)) = V^n(S_0(x)),\ \text{if } x \in A,\ 0 \leq n < N.$$

Then $S \in \mathrm{Aut}(X,\mu)$ and $SUS^{-1} = V$.

Now given aperiodic T, W and $\epsilon > 0$, find periodic U, V of large enough period N such that $\delta_u(T,U) < \epsilon, \delta_u(W,V) < \epsilon$ and find $S \in \mathrm{Aut}(X,\mu)$ with $SUS^{-1} = V$. Then as δ_u is 2-sided invariant,

$$\delta_u(STS^{-1}, W) \leq \delta_u(STS^{-1}, SUS^{-1}) + \delta_u(V, W) < 2\epsilon,$$

so we are done.

For the last assertion, note that if T is not aperiodic, then for some $n \neq 0, \epsilon < 1, \delta_u(T^n, 1) \leq \epsilon$ and so $\delta_u((STS^{-1})^n, 1) \leq \epsilon$ for every $S \in \mathrm{Aut}(X,\mu)$. □

On the other hand every conjugacy class is small in the sense of category. In fact one has a stronger conclusion concerning unitary (or spectral) equivalence classes. Consider $\mathrm{Aut}(X,\mu)$ as a closed subgroup of $U(L^2(X,\mu))$ via the identification $T \mapsto U_T$. Then we say that $T, S \in \mathrm{Aut}(X,\mu)$ are *unitarily equivalent* or *spectrally equivalent* if they are conjugate in $U(L^2(X,\mu))$. This is clearly a coarser equivalence relation than conjugacy in $\mathrm{Aut}(X,\mu)$. One now has:

Theorem 2.5 (Rokhlin). *Every unitary equivalence class in* $\mathrm{Aut}(X,\mu)$ *is meager in the weak topology. In particular, this is true for every conjugacy class in* $\mathrm{Aut}(X,\mu)$.

Proof (Hjorth). We work below in the weak topology. Fix a set $A \subseteq X$ with $\mu(A) = \frac{1}{2}$. We claim that the open set

$$\{(S,T) : \exists n[\mu(S^n(A)\Delta A) < \frac{1}{400} \text{ and } \mu(T^n(A)\Delta A) > \frac{1}{9}]\}$$

which is equal to

$$\{(S,T) : \exists n[\|U_S^n(\chi_A) - \chi_A\| < \frac{1}{20} \text{ and } \|U_T^n(\chi_A) - \chi_A\| > \frac{1}{3}]\}$$

is dense in $\mathrm{Aut}(X,\mu)^2$. To see this, notice that by 2.4 the set of mixing $T \in \mathrm{Aut}(X,\mu)$ is dense in $\mathrm{Aut}(X,\mu)$. So, given any $(S_0, T_0) \in \mathrm{Aut}(X,\mu)^2$, we can find mixing T as close as we want to T_0. Then $\mu(T^n(A) \setminus A) \to \mu(A)\mu(X \setminus A) = \frac{1}{4}$, so for some N and all $n \geq N, \mu(T^n(A)\Delta A) > \frac{1}{9}$. On the other hand, by 2.1 we can find periodic S of period $\geq N$ as close as we want to S_0. For such S, if S has period $n, \mu(S^n(A)\Delta A) = 0$, and we are done. (Notice that it would be enough here to take T weak mixing, since then there would be a set $A \subseteq \mathbb{N}$ of density 1 such that $\mu(T^n(A)\Delta A)) > \frac{1}{9}, \forall n \in A$ (see, e.g., Bergelson-Gorodnik [BGo], 1.1).)

Now, by 2.4, if a unitary equivalence class is not meager it will have to be comeager. So assume the unitary equivalence class of some T_0 is comeager, towards a contradiction. Then for comeager many $T \in \mathrm{Aut}(X,\mu)$, there is $U \in U(L^2(X,\mu))$ with $UU_TU^{-1} = U_{T_0}$. By the Jankov-von Neumann Selection Theorem, we can find a Borel function $f : \mathrm{Aut}(X,\mu) \to U(L^2(X,\mu))$ so that, letting $f_T = f(T)$,

$$\forall^* T(f_T U_T f_T^{-1} = U_{T_0}),$$

where $\forall^* T$ means "for comeager many T." Fix a dense set $\{\xi_m\} \subseteq L^2(X,\mu)$. Then $\bigcup_m \{T : \|f_T(\chi_A) - \xi_m\| < 1/20\} = \mathrm{Aut}(X,\mu)$. So there is an open nonempty set $W \subseteq \mathrm{Aut}(X,\mu)$ and some m so that, putting $\xi_m = \xi$, we have

$$\forall^* T \in W \left(\|f_T(\chi_A) - \xi\| < \frac{1}{20} \right).$$

We can then find $S, T \in W$ and n such that

$$\|U_S^n(\chi_A) - \chi_A\| < \frac{1}{20},\ \|U_T^n(\chi_A) - \chi_A\| > \frac{1}{3}$$

and $f_S U_S f_S^{-1} = f_T U_T f_T^{-1} = U_{T_0}$,

$$\|f_S(\chi_A) - \xi\| < \frac{1}{20},\ \|f_T(\chi_A) - \xi\| < \frac{1}{20}.$$

Then if $f_S(\chi_A) = \xi_1, f_T(\chi_A) = \xi_2$, we have

$$\|U_{T_0}^n(\xi_1) - \xi_1\| < \frac{1}{20},\ \|U_{T_0}^n(\xi_2) - \xi_2\| > \frac{1}{3},$$

and $\|\xi_1 - \xi\|, \|\xi_2 - \xi\| < \frac{1}{20}$, so

$$\|U_{T_0}^n(\xi) - \xi\| \leq \|U_{T_0}^n(\xi) - U_{T_0}^n(\xi_1)\| + \|U_{T_0}^n(\xi_1) - \xi_1\| + \|\xi_1 - \xi\|,$$

which is smaller than $\frac{3}{20}$, while also

$$\|U_{T_0}^n(\xi) - \xi\| \geq \|U_{T_0}^n(\xi_2) - \xi_2\| - \|U_{T_0}^n(\xi_2) - U_{T_0}^n(\xi)\| - \|\xi - \xi_2\|,$$

which is bigger than $\frac{7}{30}$, a contradiction. □

Remark. One can employ a similar strategy to give another proof of the well-known fact (see Nadkarni [Na], Ch. 8) that in the unitary group $U(H)$ of a separable infinite dimensional Hilbert space H (with the weak, equivalently strong, topology) every conjugacy class is meager.

Fix a unit vector $v \in H$. As in the proof of 2.5, it is enough to show that for any positive $\epsilon, \delta < 1$ the open set

$$W = \{(S,T) \in U(H)^2 : \exists n(\|S^n(v) - v\| < \epsilon \text{ and } \|T^n(v) - v\| > \delta)\}$$

is dense in $U(H)^2$. Fixing a basis $\{b_i\}_{i=1}^\infty$ for H, with $v = b_1$, we can view the unitary group $U(m)$ of $\mathbb{C}^m$ as the unitary group of the subspace $\langle b_i \rangle_{i=1}^m$ and it is easy to see that $\bigcup_m U(m)$ is dense in $U(H)$. So it is enough to show that given any m and $S_0, T_0 \in U(m)$, there are $S, T \in U(m)$ as close we want to S_0, T_0 such that for some n, $\|S^n - I\| < \epsilon$ and $\|T^n + I\| < 2 - \delta$, where I is the identity of $U(m)$ and $\|\cdot\|$ refers here to the norm of operators on $\mathbb{C}^n$. For this it is enough to show again that given any $U \in U(m)$ and $\rho > 0$, there is N such that for $n \geq N$, there are $V, W \in U(m)$ with $\|U - V\|, \|U - W\| < \rho$ and $V^n = I, W^n = -I$.

Since any $U_0 \in U(m)$ is conjugate in $U(m)$ to some $U_1 \in U(m)$ which is diagonal for an orthonormal basis of $\mathbb{C}^m$, i.e., for such a basis $\{e_1, \dots e_m\}$ we have $U_1(e_i) = \lambda_i e_i$, where $\lambda_i \in \mathbb{T}$, we can assume that U is already of that form. Given $\rho > 0$, there is $\sigma > 0$, such that if $P \in U(m)$ satisfies $P(e_i) = \mu_i e_i$, where $|\mu_i - \lambda_i| < \sigma, 1 \leq i \leq m$, then $\|U - P\| < \rho$. Now we can find large enough N, such that if $n \geq N$, there are nth roots of 1, say $\lambda'_1, \dots, \lambda'_m$, with $|\lambda'_i - \lambda_i| < \sigma$, and nth roots of -1, say $\lambda''_1, \dots, \lambda''_m$, with $|\lambda''_i - \lambda_i| < \sigma$. Then if $V(e_i) = \lambda'_i e_i, W(e_i) = \lambda''_i e_i, V, W$ clearly work.

Addendum. Rosendal [Ro] has recently found the following simple proof that all unitary equivalence classes in $\mathrm{Aut}(X, \mu)$ are meager in the weak topology.

For each infinite set $I \subseteq \mathbb{N}$, let $A(I) = \{T \in \mathrm{Aut}(X, \mu) : \exists i \in I(T^i = 1)\}$. By 2.2 and 2.3, $A(I)$ is dense in $(\mathrm{Aut}(X, \mu), w)$. Let $V_0 \supseteq V_1 \supseteq \dots$ be a basis of open nbhds of 1 and consider the set $B(I, k) = \{T \in \mathrm{Aut}(X, \mu) : \exists i \in I(i > k \ \& \ T^i \in V_k\}$. It contains $A(I \setminus \{0, \dots, k\})$, so it is open dense. Thus

$$C(I) = \bigcap_k B(I, k) = \{T : \exists \{i_n\} \subseteq I(T^{i_n} \to 1)\}$$

is comeager and clearly invariant under unitary equivalence. If a unitary equivalence class C is non-meager, it is contained in all $C(I), I \subseteq \mathbb{N}$ infinite. It follows that if $T \in C$, then $T^n \to 1$, so $T = 1$, a contradiction.

Notice that this proof shows that in any Polish group G in which the sets $A(I) = \{g \in G : \exists i \in I(g^i = 1)\}$ are dense for all infinite $I \subseteq \mathbb{N}$, every conjugacy class is meager. Another group that has this property is $U(H)$ (see the preceding remark) and this gives a simple proof that conjugacy classes in $U(H)$ are meager.

Denote by ERG the set of ergodic $T \in \mathrm{Aut}(X, \mu)$.

Theorem 2.6 (Halmos [Ha]). ERG *is dense* G_δ *in* $(\mathrm{Aut}(X, \mu), w)$.

Proof. Take $X = 2^{\mathbb{N}}$. For $T \in \mathrm{Aut}(X, \mu)$, define $f_T : X \to X^{\mathbb{Z}}$ by

$$f_T(x)_n = T^{-n}(x).$$

Then if S is the (left) shift action on $X^{\mathbb{Z}}$,

$$S((x_n)) = (x_{n-1}),$$

clearly $f_T(T^n(x)) = S^n(f_T(x))$. So if $\nu_T = (f_T)_* \mu, \nu_T$ is shift-invariant. We will verify that $T \mapsto \nu_T$ is continuous from $(\mathrm{Aut}(X, \mu), w)$ to the space of shift-invariant probability measures on $X^{\mathbb{Z}}$, which is a convex compact set in the usual weak*-topology of probability measures on the compact space $X^{\mathbb{Z}}$. Then

$$T \in \mathrm{ERG} \Leftrightarrow \nu_T \text{ is an ergodic, shift-invariant measure.}$$

But the ergodic shift-invariant measures are exactly the extreme points of the convex, compact set of shift-invariant measures, so they form a G_δ set and thus ERG is G_δ.

To check the continuity of $T \mapsto \nu_T$ we need to verify that for some uniformly dense $\mathcal{D} \subseteq C(X^{\mathbb{Z}})$, and any $f \in \mathcal{D}, T \mapsto \int f d\nu_T$ is continuous. By Stone-Weierstrass we can take $\mathcal{D}$ to be the complex algebra generated by the functions $(x_i)_{i \in \mathbb{Z}} \mapsto \chi_A(x_n)$, for $n \in \mathbb{Z}, A \subseteq X$ clopen. Thus it is enough to check that for each finite $\{n_1, \ldots, n_k\} \subseteq \mathbb{Z}$ and clopen sets $\{A_1, \ldots, A_k\}$ the function

$$\begin{aligned} T \mapsto & \int \prod_{i=1}^{k} \chi_{A_i}(x_{n_i}) d\nu_T((x_n)) \\ &= \int \prod_{i=1}^{k} \chi_{A_i}(T^{-n_i}(x)) d\mu(x) \\ &= \int \prod_{i=1}^{k} \chi_{T^{n_i}(A_i)}(x) d\mu(x) \\ &= \int \chi_{\bigcap_{i=1}^{k} T^{n_i}(A_i)} d\mu \\ &= \mu(\bigcap_{i=1}^{k} T^{n_i}(A_i)) \end{aligned}$$

is continuous, which is clear. □

Concerning stronger notions of ergodicity, the set of *weak mixing* transformations, WMIX, is also dense G_δ (Halmos [Ha]; see 12.1 for a more general statement) but the set of *mild mixing* transformations, MMIX, and the set of (strong) *mixing* transformations, MIX, are meager (Rokhlin for MIX; see Katok-Thouvenot [KTh], 5.48 and Nadkarni [Na], Chapter 8).

(B) Recall that if $T \in \mathrm{Aut}(X, \mu)$ and $U_T \in U(L^2(X, \mu))$ is the corresponding unitary operator, then ERG, MMIX, WMIX, MIX are characterized as follows in terms of U_T, where we put below $L^2_0(X, \mu) = \{f \in L^2(X, \mu) : \int f d\mu = 0\} = \mathbb{C}^\perp$ (the orthogonal of the constant functions):

$$\begin{aligned} T \in \mathrm{ERG} &\Leftrightarrow U_T(f) \neq f, \forall f \in L^2_0(X, \mu) \setminus \{0\}, \\ T \in \mathrm{WMIX} &\Leftrightarrow U_T(f) \neq \lambda f, \forall \lambda \in \mathbb{T} \forall f \in L^2_0(X, \mu) \setminus \{0\}, \\ T \in \mathrm{MMIX} &\Leftrightarrow U_T^{n_k}(f) \not\to f, \forall n_k \to \infty, \forall f \in L^2_0(X, \mu) \setminus \{0\}, \\ T \in \mathrm{MIX} &\Leftrightarrow \langle U^n(f), f \rangle \to 0, \forall f \in L^2_0(X, \mu). \end{aligned}$$

We have MIX $\subseteq$ MMIX $\subseteq$ WMIX $\subseteq$ ERG and these are proper inclusions. (Note that if for any $T \in \mathrm{Aut}(X, \mu)$ we denote by κ^0_T the Koopman representation of $\mathbb{Z}$ on $L^2_0(X, \mu)$ induced by U_T, then T is in ERG, etc., iff κ^0_T is ergodic, etc., according to the definition in Appendix H.)

Also denote by σ^0_T the maximal spectral type of $U_T | L^2_0(X, \mu)$ (thus σ^0_T is uniquely defined up to measure equivalence); see Appendix F (where we take $\Delta = \mathbb{Z}, H = L^2_0(X, \mu), \pi(n) = U^n_T | L^2_0(X, \mu))$. Note that σ^0_T can be chosen so that the map $T \mapsto \sigma^0_T$ is continuous from $(\mathrm{Aut}(X, \mu), w)$ into $P(\mathbb{T})$, the space of probability measures on $\mathbb{T}$ with the weak*-topology. Then we have

$$T \in \mathrm{ERG} \Leftrightarrow \sigma_T^0(\{1\}) = 0,$$
$$T \in \mathrm{WMIX} \Leftrightarrow \sigma_T^0 \text{ is non-atomic},$$
$$T \in \mathrm{MMIX} \Leftrightarrow \sigma_T^0 \in D^\perp,$$
$$T \in \mathrm{MIX} \Leftrightarrow \sigma_T^0 \in R,$$

where R is the class of *Rajchman measures* in $P(\mathbb{T})$, i.e., those measures $\mu \in P(\mathbb{T})$ such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, D is the class of closed *Dirichlet sets* in $\mathbb{T}$, i.e., the class of closed $E \subseteq \mathbb{T}$ such that there is $n_k \to \infty$ with $z^{n_k} \to 1$ uniformly for $z \in E$, and finally $D^\perp$ is the class of all $\mu \in P(\mathbb{T})$ that annihilate all Dirichlet sets. All the preceding characterizations in terms of U_T, σ_T^0 can be found, for example, in Queffélec [Qu]. The reader should note that in Queffeléc [Qu], III.21 one finds the characterization: $T \in \mathrm{MMIX} \Leftrightarrow \sigma_T^0 \in D^\perp$ in the form $T \in \mathrm{MMIX} \Leftrightarrow \sigma_T^0 \in \mathcal{L}_I$, where $\mathcal{L}_I$ is a certain class of measures defined there. A proof that $\mathcal{L}_I = D^\perp$ can be found in Proposition 3, p. 212 and Proposition 9, p. 243 of Host-Méla-Parreau [HMP]. For more about Dirichlet sets, see Lindahl–Poulsen [LP], Ch.1.

For each $\sigma \in P(\mathbb{T})$ which is symmetric (i.e., invariant under conjugation), $\hat{\sigma}(n) = \int z^{-n} d\sigma(z)$ is a positive-definite real function and let T_σ be the Gaussian shift on $\mathbb{R}^\mathbb{Z}$ corresponding to $\hat{\sigma}$ (see Appendix C). It is known (see Cornfeld-Fomin-Sinai [CFS], 14.3, Theorem 1) that the maximal spectral type of $U_{T_\sigma}|\mathbb{C}^\perp$ is $\sigma_\infty = \sum_{n=1}^\infty \frac{1}{2^n}\sigma^{*n}$, where $\sigma^{*n} = \sigma * \cdots * \sigma$ is the n-fold convolution of σ. We also have (see Cornfeld-Fomin-Sinai [CFS], 14.2) and Queffélec [Qu], III. 21):

$$\sigma \text{ is non-atomic} \Leftrightarrow T_\sigma \in \mathrm{ERG} \Leftrightarrow T_\sigma \in \mathrm{WMIX},$$
$$\sigma \in D^\perp \Leftrightarrow T_\sigma \in \mathrm{MMIX},$$
$$\sigma \in R \Leftrightarrow T_\sigma \in \mathrm{MIX}.$$

For the equivalence $\sigma \in D^\perp \Leftrightarrow T_\sigma \in \mathrm{MMIX}$ one needs to notice that $\sigma \in D^\perp \Rightarrow \sigma_\infty \in D^\perp$, which follows from the fact that $D^\perp$ is closed under convolution by any measure, since D is closed under translations (see Lindahl–Poulsen [LP], p.4). We already noted that ERG, WMIX are G_δ in $(\mathrm{Aut}(X,\mu), w)$ and it is easy to see that MIX is $\mathbf{\Pi}_3^0$. However we have the following result.

Theorem 2.7 (Kaufman, Kechris). *The set of mild mixing transformations in* $\mathrm{Aut}(X,\mu)$ *is co-analytic but not Borel in the weak topology of* $\mathrm{Aut}(X,\mu)$.

Proof. It is clear that MMIX is co-analytic. The map $\sigma \mapsto T_\sigma \in \mathrm{Aut}(X,\mu)$ is Borel and $D^\perp$ restricted to symmetric measures is known to be non-Borel (Kechris-Lyons [KLy], Host-Louveau-Parreau). □

Remark. Recall that a subset A of $\mathbb{N} \setminus \{0\}$ is an IP-*set* if there are $p_1, p_2, \dots \in \mathbb{N} \setminus \{0\}$ such that $A = \{p_{i_1} + \cdots + p_{i_k} : i_1 < \cdots i_k\}$. The filter

IP* consists of all $A \subseteq \mathbb{N} \setminus \{0\}$ that intersect every IP-set (see Furstenberg [Fur]). It is shown in [Fur], 9.22, that $T \in \mathrm{Aut}(X,\mu)$ is mild mixing iff for any two Borel sets $A, B \subseteq X$, $\mu(A \cap T^{-n}(B)) \to \mu(A)\mu(B)$ in the filter IP*. It is clear that A, B can be restricted to any countable dense subset of the measure algebra. It follows that IP*, which is clearly a co-analytic set in the compact metric space of subsets of $\mathbb{N} \setminus \{0\}$, is not Borel.

In Kechris-Lyons [KLy] a canonical $\mathbf{\Pi}^1_1$-rank on $D^\perp$ is defined and used to prove, by a boundedness argument, the non-Borelness of $D^\perp$. It might be interesting to define a canonical $\mathbf{\Pi}^1_1$-rank on MMIX, as this would probably give an interesting transfinite hierarchy of progressively "milder" mild mixing notions. One could even wonder whether mixing would occupy exactly the lowest level of this hierarchy.

(C) We next show that $\mathrm{Aut}(X,\mu)$ is contractible.

Theorem 2.8 (Keane [Kea]). *The group* $\mathrm{Aut}(X,\mu)$ *is contractible in both the weak and the uniform topology.*

Proof. For each $A \in \mathrm{MALG}_\mu$, $T \in \mathrm{Aut}(X,\mu)$, define the *induced transformation* $T_A \in \mathrm{Aut}(X,\mu)$ as follows: Let $B_0 = X \setminus A$, $B_1 = A \cap T^{-1}(A), \dots, B_n = A \cap T^{-1}(B_0) \cap \cdots \cap T^{-(n-1)}(B_0) \cap T^{-n}(A), n \geq 2$. The Poincaré Recurrence Theorem guarantees that $\{B_n\}$ is a partition of X (a.e.). Let $T_A(x) = T^n(x)$, if $x \in B_n$. Then $(T, A) \mapsto T_A$ is continuous from $(\mathrm{Aut}(X,\mu), u) \times \mathrm{MALG}_\mu \to (\mathrm{Aut}(X,\mu), u)$ and also from $(\mathrm{Aut}(X,\mu), w) \times \mathrm{MALG}_\mu \to (\mathrm{Aut}(X,\mu), w)$. Assume now $X = [0,1]$, $\mu =$ Lebesgue measure, and define $\varphi : [0,1] \times \mathrm{Aut}(X,\mu) \to \mathrm{Aut}(X,\mu)$ by $\varphi(\lambda, T) = T_{[\lambda,1]}$. Then this is a contraction to the identity of $\mathrm{Aut}(X,\mu)$. □

For the weak topology a complete topological characterization has been obtained.

Theorem 2.9 (Nhu [Nh]). *The space* $(\mathrm{Aut}(X,\mu), w)$ *is homeomorphic to* ℓ^2.

(D) Finally we mention a number of algebraic properties of $\mathrm{Aut}(X,\mu)$.

Theorem 2.10 (Fathi [Fa], Eigen [Ei2], Ryzhikov [Ry1]). *The group* $\mathrm{Aut}(X,\mu)$ *has the following properties:*

(i) Every element is a commutator and the product of 3 involutions.

(ii) It is a simple group.

(iii) Every automorphism is inner.

Proofs of (ii), (iii) and a somewhat weaker version of (i) are contained in the arguments given in the course of the proof of Theorem 4.1 below

Comments. For 2.5 see Nadkarni [Na], §8 and references contained therein. For 2.10, see Choksi-Prasad [CP] for some generalizations.

3. Full groups of equivalence relations

(A) Let (X, μ) be a standard measure space and E a Borel equivalence relation on X. We say that E is a *countable equivalence relation* if every E-class is countable. Feldman-Moore [FM] have shown that E is countable iff there is a countable (discrete) group Γ acting in a Borel way on X such that if $(\gamma, x) \mapsto \gamma \cdot x$ denotes the action and we let

$$E_\Gamma^X = \{(x, y) : \exists \gamma \in \Gamma (\gamma \cdot x = y)\}$$

be the induced equivalence relation, then $E = E_\Gamma^X$. We say that E is a *measure preserving equivalence relation* if the action of Γ is measure preserving, i.e., $x \mapsto \gamma \cdot x$ is measure preserving for each $\gamma \in \Gamma$. This condition is independent of Γ and the action that induces E and is equivalent to the condition that every Borel automorphism T of X for which $T(x)Ex, \forall x$, is measure preserving.

For any Borel E we define its *full group* or *inner automorphism group*, $[E]$, by

$$[E] = \{T \in \mathrm{Aut}(X, \mu) : T(x)Ex,\ \mu\text{-a.e.}(x)\}.$$

It is easy to check that this is a closed subgroup of $\mathrm{Aut}(X, \mu)$ in the uniform topology.

Convention. *Below we will assume that equivalence relations E are countable and measure preserving, unless otherwise explicitly stated.*

Recall that E is an *ergodic equivalence relation* if every E-invariant Borel set is null or co-null. We call E a *finite equivalence relation* or a *periodic equivalence relation* if (almost) all its classes are finite and an *aperiodic equivalence relation* if (almost) all its classes are infinite.

Proposition 3.1. *E is ergodic iff $[E]$ is dense in* $(\mathrm{Aut}(X, \mu), w)$.

Proof. Assume E is ergodic. Then if A, B are Borel sets of the same measure, there is $T \in [E]$ with $T(A) = B$.

Now let $S \in \mathrm{Aut}(X, \mu)$. A nbhd basis for S is given by the sets of the form $\{T : \forall i \leq n (d_\mu(T(A_i), S(A_i)) < \epsilon)\}$, where $\epsilon > 0$, and $A_1, \dots, A_n$ is a Borel partition of X. Find $T_1, \dots, T_n \in [E]$ such that $T_i(A_i) = S(A_i)$. Then $T = \bigcup_{i=1}^n T_i | A_i$ is in $[E]$ and clearly $T(A_i) = S(A_i),\ i = 1, \dots, n$.

If E is not ergodic, there is $A \in \mathrm{MALG}_\mu$ with $0 < \mu(A) < 1$ which is $[E]$-invariant. If $[E]$ was weakly dense in $\mathrm{Aut}(X, \mu)$, A would be T-invariant for any $T \in \mathrm{Aut}(X, \mu)$, which is absurd. □

T.-J. Wei [We] has shown in fact that $[E]$ is closed in w iff E is a finite equivalence relation. Otherwise, it is $\mathbf{\Pi}_3^0$-complete in w and its closure in the weak topology is equal to $[F]$, where F is the (not necessarily countable) equivalence relation corresponding to the ergodic decomposition of E. Also clearly $[E]$ is a meager subset of $(\mathrm{Aut}(X, \mu), w)$, since it is a Borel non-open subgroup of $(\mathrm{Aut}(X, \mu), w)$.

Note now the following basic fact about the uniform topology of $[E]$.

Proposition 3.2. *In the uniform topology,* $[E]$ *is separable, thus* $([E], u)$ *is Polish.*

Proof. Since E is a countable equivalence relation, there is a countable group $\Gamma \le \mathrm{Aut}(X, \mu)$ such that $E = E_\Gamma^X =$ the equivalence relation induced by Γ. Thus $T \in [E]$ iff there is a Borel partition $\{A_n\}_{n=1}^\infty$ of X and $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$ such that $T|A_n = \gamma_n|A_n$. Fix also a countable Boolean algebra $\mathcal{A}$ of Borel sets closed under the Γ-action and dense in the measure algebra.

Given $C_1, \dots, C_N \in \mathcal{A}$ and $\gamma_1, \dots, \gamma_N \in \Gamma$ such that $C_1, \dots, C_N$ are pairwise disjoint and $\gamma_1(C_1), \dots, \gamma_N(C_N)$ are pairwise disjoint, let $C_0 = X \setminus \bigcup_{i \le N} C_i, D_0 = X \setminus \bigcup_{i \le N} \gamma_i(C_i)$, so that $\mu(C_0) = D_0$, and pick $T_{C_0, D_0} \in \mathrm{Aut}(X, \mu)$ with $T(C_0) = D_0$. Put finally $S_{\vec{C}, \vec{\gamma}} = T_{C_0, D_0}|C_0 \cup \bigcup_{i \le N} \gamma_i | C_i$. Clearly there are only countably many $S_{\vec{C}, \vec{\gamma}}$. We will show that for any $T \in [E]$ and $\epsilon > 0$ there are $\vec{C}, \vec{\gamma}$ such that $\delta_u(T, S_{\vec{C}, \vec{\gamma}}) < \epsilon$. It follows then that $[E]$ is separable (being contained in the uniform closure of the $S_{\vec{C}, \vec{\gamma}}$).

So fix $T \in [E], \epsilon > 0$, a Borel partition $\{A_n\}_{n=1}^\infty$ of X and $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$ such that $T|A_n = \gamma_n|A_n, \forall n \ge 1$. For each $\delta_1, \delta_2 > 0$, choose $N = N(\delta_1)$ large enough so that $\sum_{i > N} \mu(A_i) < \delta_1$, and then $B_1, \dots, B_N \in \mathcal{A}$ with $\mu(A_i \Delta B_i) < \delta_2$. Let for $1 \le i \le N$,

$$C_i = \{x \in B_i : x \notin \bigcup_{j<i} B_j \ \&\ \gamma_i(x) \notin \bigcup_{j<i} \gamma_j(C_j)\},$$

so that $\{C_i\}_{i \le N} \subseteq \mathcal{A}$ are pairwise disjoint and so are $\{\gamma_i(C_i)\}_{i \le N} \subseteq \mathcal{A}$. Then if δ_1, δ_2 are chosen small enough, $\delta_u(T, S_{\vec{C}, \vec{\gamma}}) < \epsilon$. □

Since the identity is a continuous map from the Polish group $([E], u)$ onto the Borel subgroup $[E]$ of $(\mathrm{Aut}(X, \mu), w)$, it follows that $[E]$ is a Polishable subgroup of $(\mathrm{Aut}(X, \mu), w)$ with the unique corresponding Polish topology being the uniform topology.

(B) Some of the basic facts we mentioned earlier for $\mathrm{Aut}(X, \mu)$ "localize" to $[E]$.

First the proof of Rokhlin's Lemma actually shows that the Uniform Approximation Theorem is true in each $[E]$.

Theorem 3.3 (Uniform Approximation Theorem for [*E*]). *If* $T \in [E]$ *is aperiodic, then for each* $N \ge 1, \epsilon > 0$ *there is a periodic* $S \in [E]$ *of period* N *such that* $\delta_u(S, T) \le \frac{1}{N} + \epsilon$.

From this we also have the following, in case E is also ergodic.

Theorem 3.4 (Conjugacy Lemma for [*E*]). *If* E *is ergodic, and* $T \in [E]$ *is aperiodic,* $\{STS^{-1} : S \in [E]\}$ *is uniformly dense in* $\mathrm{APER} \cap [E]$ *(which is a closed subset of* $([E], u)$*).*

Proof. In the proof of 2.4 note that, since E is ergodic, there is $S_0 \in [E]$ with $S_0(A) = B$. □

We also note another well-known basic fact.

Theorem 3.5. *If E is ergodic (resp., aperiodic), then there is $T \in [E]$ which is ergodic (resp., aperiodic).*

Proof. (i) We first give the proof for the aperiodic case, which works in the pure Borel context (with no measures present). The proof below is due to Dougherty.

Let E be a countable aperiodic Borel equivalence relation on a standard Borel space X. Let $<$ be a Borel linear order on X and let a countable group Γ act in a Borel way on X so that $E = E_\Gamma^X$. Put $\Gamma = \{\gamma_n\}$. Let $A = \{x : [x]_E \text{ has a largest element in } <\}$. Then $E|A$ admits a Borel transversal, so clearly there is an aperiodic Borel automorphism of A, say T_A, with $T_A(x)Ex, \forall x \in A$. Let now $B = X \setminus A$. Define $U : B \to B$ by $U(x) = \gamma_n(x)$, where n is least with $\gamma_n(x) > x$. Consider then the equivalence relation on B:

$$xE_U y \Leftrightarrow \exists m \exists n (U^n(x) = U^m(y)).$$

There is a Borel automorphism of B, say T_B, that induces E_U (see Dougherty-Jackson-Kechris [DJK], 8.2). Clearly $T_B(x)Ex, \forall x \in B$, and T_B is aperiodic. So $T = T_A \cup T_B$ works.

(ii) For the ergodic case, see Zimmer [Zi1], 9.3.2 or Kechris [Kec1], 5.66. Alternatively, one can go over the proof of Dye's Theorem, 3.13 below, given in Kechris-Miller [KM], 7.13, and notice that if condition (4) is omitted in that argument, one obtains a proof that every ergodic E contains an ergodic subequivalence relation isomorphic to E_0, defined before 3.8 below, which is thus generated by an ergodic automorphism. □

As an immediate consequence, we obtain the next result.

Theorem 3.6. *If E is ergodic, then* $\mathrm{ERG} \cap [E]$ *is dense G_δ in* $(\mathrm{APER} \cap [E], u)$.

Proof. Since ERG is G_δ in $w \subseteq u$, it is clearly G_δ in u. Density follows from the two preceding results. □

Similarly (using also 3.14 below) one can show the same fact for the set of weak mixing $T \in [E]$.

It will be useful, given a countable group $\Gamma \leq \mathrm{Aut}(X, \mu)$ and letting $E = E_\Gamma^X = \{(x, y) : \exists \gamma \in \Gamma(\gamma(x) = y)\}$, to have a criterion for a countable subgroup $\Delta \leq [E]$ to be dense in $([E], u)$. Note that if $\Delta \leq [E]$ is dense in $([E], u)$, then $E = E_\Delta^X$. (However, if $E = E_\Gamma^X, \Gamma$ may not be dense in $([E], u)$.)

Proposition 3.7. *Suppose $\Gamma \leq \mathrm{Aut}(X, \mu)$ is a countable group and $E = E_\Gamma^X$. Let also $\mathcal{A} \subseteq \mathrm{MALG}_\mu$ be a countable Boolean algebra dense in MALG_μ and closed under Γ. Let $\Delta \leq [E]$ be a countable subgroup of $[E]$. Then Δ is dense in $([E], u)$ provided that for every $\epsilon > 0$ and every $A_0, \dots, A_{n-1} \in \mathcal{A}, \gamma_0, \dots, \gamma_{n-1} \in \Gamma$ with $\{A_i\}_{i=0}^{n-1}, \{\gamma_i(A_n)\}_{i=0}^{n-1}$ each pairwise disjoint, there is $\delta \in \Delta$ such that $\mu(\{x \in A_i : \gamma_i(x) \neq \delta(x)\}) < \epsilon, i = 0, \dots, n-1$.*

The proof is similar to that of 3.2, so we omit it. Notice that this condition is also necessary if E is ergodic.

This has the following consequence; below E_0 denotes the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists m \forall n \geq m(x_n = y_n).$$

Proposition 3.8. *The group of dyadic permutations of* $2^{\mathbb{N}}$ *is uniformly dense in* $[E_0]$.

Proof. We use 3.7. Take $\Gamma = \{\text{dyadic permutations}\}$, $\mathcal{A} =$ the algebra of clopen sets in $2^{\mathbb{N}}$. Thus $\mathcal{A}$ consists of all sets of the form

$$\bigcup_{s \in I} N_s, \ I \subseteq 2^n \text{ for some } n.$$

Then $E_{\Gamma}^{2^{\mathbb{N}}} = E_0$ and $\mathcal{A}$ is closed under Γ. To verify 3.7 for $\Delta = \Gamma$ take $\{A_i\}_{i<n} \subseteq \mathcal{A}$, $\{\gamma_i\}_{i<n} \subseteq \Gamma$ with $\{A_i\}_{i<n}, \{\gamma_i(A_i)\}_{i<n}$ both pairwise disjoint. We can assume that for some large enough N,

$$A_i = \bigcup_{s \in I_i} N_s, \ I_i \subseteq 2^N,$$

$$\gamma_i(A_i) = \bigcup_{s \in J_i} N_s, \ J_i \subseteq 2^N,$$

and γ_i is a dyadic permutation of rank N, $i = 0, \ldots, n-1$. Thus each of $\{I_i\}_{i<n}, \{J_i\}_{i<n}$ is pairwise disjoint and if $\gamma_i(s\hat{}x) = \pi_i(s)\hat{}x, \pi_i$ a permutation of 2^N, then $\pi_i(I_i) = J_i, i = 0, \ldots, n-1$. So there is a permutation π of 2^N with $\pi|I_i = \pi_i|I_i$. Thus if $\gamma \in \Gamma$ is defined by $\gamma(s\hat{}x) = \pi(s)\hat{}x, s \in 2^N$, clearly $\gamma|A_i = \gamma_i|A_i, i = 0, \ldots, n-1$ and we are done. □

Remark. In connection with 3.3 and 3.8, Ben Miller has shown that if X is a zero-dimensional Polish space, μ a non-atomic Borel probability measure on X, Γ a countable group acting by homeomorphisms on $X, E = E_{\Gamma}^X$ the equivalence relation generated by this action, and $[E]_C$ the group of all homeomorphisms f of X such that $\forall x \in X(f(x)Ex)$, then the periodic (i.e., having finite orbits) homeomorphisms in $[E]_C$ are uniformly dense in $[E]$.

(C) We next verify that a generic pair in APER$\cap[E]$ generates a free group.

Theorem 3.9. *The set*

$$\{(S,T) \in (\text{APER} \cap [E])^2 : S, T \text{ generate a free group}\}$$

is dense G_δ *in* $(\text{APER}\cap[E], u)^2$.

Proof. It is clearly G_δ. To prove density it is enough, by the Baire Category Theorem, to verify that for each reduced word $w(x,y)$,

$$\{(S,T) \in (\text{APER} \cap [E])^2 : w(S,T) \neq 1\}$$

is uniformly dense in $(\text{APER}\cap[E])^2$.

Otherwise, find $(S_0, T_0) \in (\text{APER} \cap [E])^2, \epsilon > 0$ such that for every $(S,T) \in (\text{APER} \cap [E])^2$ with $\delta_u(S, S_0) < \epsilon, \delta_u(T, T_0) < \epsilon$, we have $w(S,T) = 1$, i.e., $w(S,T)(x) = x, \mu$–a.e. (x). Let $w_1, w_2, \dots, w_n = w$ be the co-initial (non-$\emptyset$) subwords of w (e.g., if $w = xy^2x^{-1}, w_1 = x^{-1}, w_2 = yx^{-1}, w_3 = y^2x^{-1}, w_4 = xy^2u^{-1} = w$). Clearly $n \geq 2$.

Lemma 3.10. *If $P_1, \dots, P_k \in \text{Aut}(X, \mu)$, and on a Borel set of positive measure A and every $x \in A$ we have that $P_1(x), P_2P_1(x), \dots, P_k \cdots P_1(x)$ are distinct, then there is $B \subseteq A$ of positive measure such that the sets $P_1(B), P_2P_1(B), \dots, P_k \cdots P_1(B)$ are pairwise disjoint.*

Proof. We can assume that X is a Polish space, $A \subseteq X$ is clopen and $P_1, \dots, P_k$ are homeomorphisms. Fix a countable basis $\{U_n\}$ for X. Then every point $x \in A$ is contained in a basic open set $U_{n(x)}$ such that $P_1(U_{n(x)}), \dots, P_k \cdots P_1(U_{n(x)})$ are pairwise disjoint. As $A \subseteq \bigcup\{U_{n(x)} : x \in A\}$, for some $x \in A, B = U_{n(x)} \cap A$ has positive measure. □

For each $T \in \text{Aut}(X, \mu)$, let E_T be the equivalence relation induced by T,

$$xE_Ty \Leftrightarrow \exists n \in \mathbb{Z}(T^n(x) = y)$$

and let

$$[T] = [E_T].$$

Lemma 3.11. *Given aperiodic $T \in \text{Aut}(X, \mu)$ and a Borel set with $\mu(A) > 0$, there is a Borel set $A' \subseteq A$ with $\mu(A') > 0$ and aperiodic $T' \in [T]$ such that $T(x) = T'(x), \mu$-a.e. $x \notin A', T(x) \neq T'(x), \mu$-a.e. $x \in A'$.*

Proof. We can assume that A intersects every orbit of T and, by Poincaré recurrence, that for (almost) every $x \in A$ there are $k, \ell > 0$ with $T^k(x), T^{-\ell}(x) \in A$. We can also assume that if $B = X \setminus A$, then $\mu(B) > 0$ and $\forall x \in B \exists k, \ell > 0(T^k(x) \in B, T^{-\ell}(x) \in B)$ (since if $\mu(B) = 0$ we can take $A' = A = X, T'(x) = T^2(x)$). Then we can find $A' \subseteq A \cap [B]_T$, where $[B]_T$ denotes the T-saturation of B, such that $[A']_T = [B]_T$, so that $\mu(A') > 0, \forall x \in A' \exists k > 0 \exists \ell > 0(T^k(x), T^{-\ell}(x) \in A')$ and finally $x \in A' \Rightarrow T(x) \notin A'$. Then we put $T'(x) = T(x)$ if $x \notin A'$, and $T'(x) = T(T^k(x))$, where $k > 0$ is least with $T^k(x) \in A'$, if $x \in A'$. □

For convenience, in the rest of the argument, we will write, for any reduced word $v(x,y)$, $v^{S,T}$ instead of $v(S,T)$. Continuing the proof, let $0 < k_0 \leq n$ be least such that on a set of positive measure A_0 we have

$$x \in A_0 \Rightarrow x, w_1^{S_0,T_0}(x), \dots, w_{k_0-1}^{S_0,T_0}(x_0) \text{ are distinct and}$$
$$w_{k_0}^{S_0,T_0}(x) \in \{x, \dots, w_{k_0-1}^{S_0,T_0}(x)\}.$$

Clearly $k_0 \geq 2$ and actually

$$w_{k_0}^{S_0,T_0}(x) \in \{x, \dots, w_{k_0-2}^{S_0,T_0}(x)\}$$

(since S_0, T_0 are aperiodic). By 3.10, we can also assume that

$$A_0, w_1^{S_0,T_0}(A_0), \dots, w_{k_0-1}^{S_0,T_0}(A_0)$$

are pairwise disjoint and if the first symbol of w_{k_0} is say $S_0^{\pm 1}$ (the other case being similar), then $S_0^{\pm 1}(w_{k_0-1}^{S_0,T_0}(A_0)) = w_i^{S_0,T_0}(A_0)$, for some $0 \leq i \leq k_0 - 2$. Finally, we can assume that $\mu(A_0) < \epsilon/2$.

If the first symbol of w_{k_0} is S_0 (the other case being similar), then by 3.11, applied to the set $A = w_i^{S_0,T_0}(A_0)$ and $T = S_0^{-1}$, we can find the appropriate $T', A' \subseteq A$ and then, letting $S_0' = (T')^{-1}, A_0' = (w_i^{S_0,T_0})^{-1}(A')$ we have an aperiodic S_0' with $\delta_u(S_0, S_0') < \epsilon/2$ and $A_0' \subseteq A_0, \mu(A_0') > 0$ such that

$$x, w_1^{S_0',T_0}(x), \dots, w_{k_0-1}^{S_0',T_0}(x), w_{k_0}^{S_0',T_0}(x)$$

are distinct for $x \in A_0'$. (Note here that $\forall j < k_0, w_j^{S_0',T_0}(x) = w_j^{S_0,T_0}(x), \forall x \in A_0'$.) Since $\delta_u(S_0, S_0') < \epsilon/2$, we still have $w_n^{S_0',T_0}(x) = x$ for almost all $x \in A_0'$.

Then, if $k_0 < n$, let $0 < k_0 < k_1 \leq n$ be least such that on a set $A_1 \subseteq A_0'$ of measure $< \epsilon/4$

$$x \in A_1 \Rightarrow x, w_1^{S_0',T_0}(x), \dots, w_{k_1-1}^{S_0',T_0}(x) \text{ are distinct and}$$
$$w_{k_1}^{S_0',T_0}(x) \in \{x, w_1^{S_0',T_0}(x), \dots, w_{k_1-1}^{S_0',T_0}(x)\}$$

and repeat this process finitely many times to eventually find $\bar{S}, \bar{T}$ with $\delta_u(\bar{S}, S_0) < \epsilon, \delta_u(\bar{T}, T_0) < \epsilon$ and a set $\bar{A}$ of positive measure such that

$$x \in \bar{A} \Rightarrow x, w_1^{\bar{S},\bar{T}}(x), \dots, w_n^{\bar{S},\bar{T}}(x) \text{ are distinct},$$

which is a contradiction, as for almost all $x \in \bar{A}, w_n^{\bar{S},\bar{T}}(x) = x$. □

A similar argument shows that the set of $(S_1, \dots, S_n) \in (\text{APER} \cap [E])^n$ that generate a free group is also dense G_δ. So using also the *Mycielski, Kuratowski Theorem* (see Kechris [Kec2], 19.1) this shows that there is a Cantor set $C \subseteq \text{APER} \cap [E]$ generating a free group, so $[E]$ contains a free subgroup with continuum many generators.

(D) Note that the proof of 2.8 also shows the following.

Theorem 3.12 (Keane). *The full group $[E]$ is contractible for both the weak and the uniform topologies.*

Recently Kittrell and Tsankov [KiT] have shown that in fact $([E], u)$ is homeomorphic to ℓ^2.

(E) Recall that E is a *hyperfinite equivalence relation* if it is induced by an element of $\text{Aut}(X, \mu)$. In particular E_0 is hyperfinite. The fundamental result about hyperfinite equivalence relations is the following classical theorem of Dye. For a proof see, e.g., Kechris-Miller [KM].

Theorem 3.13 (Dye). *Let E, F be ergodic hyperfinite Borel equivalence relations. Then there is $T \in \mathrm{Aut}(X,\mu)$ such that for all x, y in a set of measure 1,*

$$xEy \Leftrightarrow T(x)FT(y).$$

Since the hyperfinite E which preserve μ and are ergodic are exactly the ones induced by ergodic $T \in \mathrm{Aut}(X,\mu)$, we have for any two ergodic $T_1, T_2 \in \mathrm{Aut}(X,\mu)$, that $[T_1], [T_2]$ are conjugate in $\mathrm{Aut}(X,\mu)$. Also from 3.5 we have the next result.

Corollary 3.14. *If E is ergodic, then for any ergodic $T \in \mathrm{Aut}(X,\mu)$, $[E]$ contains a conjugate of T, i.e., $[E]$ meets every conjugacy class in* ERG.

Thus, up to conjugacy, there is only one full group of a hyperfinite ergodic equivalence relation and it is the smallest (in terms of inclusion) among all full groups of ergodic countable equivalence relations.

(F) We conclude with an application of 3.8 due to Giordano-Pestov. Recall that a topological group G is *extremely amenable* if every continuous action of G on a (Hausdorff) compact space has a fixed point.

Theorem 3.15 (Giordano-Pestov [GP]). *Let E be an ergodic, hyperfinite equivalence relation. Then $([E], u)$ is extremely amenable.*

Corollary 3.16 (Giordano-Pestov [GP]). *The group $(\mathrm{Aut}(X,\mu), w)$ is extremely amenable.*

The corollary follows from 3.15, since, by 3.1, the identity map is a continuous embedding of $([E], u)$ into $(\mathrm{Aut}(X,\mu), w)$ with dense range.

Proof of 3.15. We take $E = E_0$. Denote by $G_n = S_{2^n}$ the finite group of permutations of 2^n, which we identify with a subgroup of $[E]$, identifying $\pi \in S_{2^n}$ with the element of $[E]$ given by $s\hat{\ }x \mapsto \pi(s)\hat{\ }x$. Clearly $G_1 \subseteq G_2 \subseteq \dots$ and $\bigcup_n G_n$ is dense in $([E_0], u)$ by 3.8.

Now the metric δ_u restricted to G_n is clearly the Hamming metric on $G_n : \delta_u(\pi, \rho) = \frac{1}{2^n}\mathrm{card}\{s \in 2^n : \pi(s) \neq \rho(s)\}$, and therefore, by a result of Maurey [Ma], the family $(G_n, \delta_u | G_n, \mu_n)$, where μ_n = counting measure on G_n, is a Lévy family and so, by Gromov-Milman [GM], $([E_0], u)$ is a Lévy group and thus extremely amenable (see, e.g., Pestov [Pe2]). □

Giordano-Pestov [GP] also prove that, conversely, if for an ergodic E the group $([E], u)$ is extremely amenable, then E is hyperfinite, so this gives a nice characterization of the hyperfiniteness of E in terms of properties of $[E]$:

Theorem 3.17 (Giordano-Pestov [GP]). *If E is an ergodic equivalence relation, then E is hyperfinite iff $([E], u)$ is extremely amenable.*

4. The Reconstruction Theorem

(A) We will prove here another beautiful result of Dye that asserts that any ergodic E is completely determined by $[E]$ as an *abstract group*. Given two countable Borel measure preserving equivalence relations E, F we say that E, F are *isomorphic*, in symbols

$$E \cong F,$$

if there is $T \in \mathrm{Aut}(X, \mu)$ such that except on a null set

$$xEy \Leftrightarrow T(x)FT(y),$$

or, equivalently, $T[E]T^{-1} = [F]$, i.e., $[E], [F]$ are conjugate in $\mathrm{Aut}(X, \mu)$.

Theorem 4.1 (Dye). *Let E, F be ergodic equivalence relations. Then the following are equivalent:*

(i) $E \cong F$.

(ii) $[E], [F]$ are conjugate in $\mathrm{Aut}(X, \mu)$.

(iii) $([E], u), ([F], u)$ are isomorphic as topological groups.

(iv) $[E], [F]$ are isomorphic as abstract groups.

Moreover, for any algebraic isomorphism $f : [E] \to [F]$, there is unique $\varphi \in \mathrm{Aut}(X, \mu)$ with $f(T) = \varphi T \varphi^{-1}, \forall T \in [E]$.

(B) In preparation for the proof, we prove some results that are also useful for other purposes.

Lemma 4.2. *Let E be ergodic. Then every element of $[E]$ is a product of 5 commutators in $[E]$.*

Proof. We will need two sublemmas.

Sublemma 4.3. *Let E be ergodic and let $T \in [E]$ be periodic. Then T is a commutator in $[E]$ and the product of two involutions in $[E]$.*

Proof. We can assume that for some $n \geq 2$ all T-orbits have exactly n elements. Let A be a Borel selector for the T-orbits. Split $A = A_1 \cup A_2$, where $\mu(A_1) = \mu(A_2) = \frac{1}{2}\mu(A)$. Let $Q \in [E]$ be an involution that sends $B_1 = \bigcup_{k=0}^{n-1} T^k(A_1)$ to $B_2 = \bigcup_{k=0}^{n-1} T^k(A_2)$ and conjugates $T|B_1$ to $T^{-1}|B_2$. Let $T_1 = T|B_1 \cup \mathrm{id}|B_2, T_2 = \mathrm{id}|B_1 \cup T|B_2$. Then $QT_2^{-1}Q = T_1$ and $T = T_1T_2 = QT_2^{-1}QT_2 = [Q, T_2^{-1}] = Q(T_2^{-1}QT_2)$. □

Sublemma 4.4. *Let E be ergodic. If $T \in [E]$, then we can write $T = ST'$, where $S \in [E], S = [U, V] = U_1V_1$ with $U, V, U_1, V_1 \in [E], \delta_u(T', 1) \leq \frac{1}{2}\delta_u(T, 1), U_1, V_1$ involutions and*

$$\mathrm{supp}(U) \cup \mathrm{supp}(V) \cup \mathrm{supp}(U_1) \cup \mathrm{supp}(V_1) \cup \mathrm{supp}(T') \subseteq \mathrm{supp}(T).$$

Proof. We can assume that T is aperiodic (since on the periodic part of T we can apply 4.3 and take $T' = 1$). We can then find a Borel complete section A for the T-orbits such that in the usual $\mathbb{Z}$-order of each T-orbit the distance between successive elements of A is at least 2, and A is unbounded in each direction in every orbit (see, e.g., [KM], 6.7). Let $T' = T_A$ be the induced transformation. So $T' \in [E]$ and $T(T')^{-1}$ is periodic, thus, by 4.3, $S = T(T')^{-1} = [U, V] = U_1 V_1$, for some $U, V, U_1, V_1 \in [E]$, U_1, V_1 involutions. Also $\mathrm{supp}(T') = A$ and $\mu(A) \leq \frac{1}{2}\mu(\mathrm{supp}(T)) = \frac{1}{2}$, as $A \cap T(A) = \emptyset$, so $\delta_u(T', 1) \leq \frac{1}{2}$. □

Fix now $T \in [E]$. All transformations below are in $[E]$. Then, by 4.4, $T = S_0 T_1$, where S_0 is a commutator and $\delta_u(T_1, 1) = \mu(\mathrm{supp}(T_1)) \leq \frac{1}{2}$. We concentrate on T_1. Split $X = X_1 \cup X_2 \cup \dots$, with $\mu(X_i) = 2^{-i}$, $\mathrm{supp}(T_1) \subseteq X_1$. Then, by 4.4 again, $T_1 = S_1 T_2'$, where S_1 is a commutator of elements with support contained in X_1, T_2' has support also contained in X_1, and $\delta_u(T_2', 1) \leq \frac{1}{4}$. Then find an involution $U_{1,2}$ supported by $X_1 \cup X_2$, with $U_{1,2}(\mathrm{supp}(T_2')) \subseteq X_2$, thus, as in general $W(\mathrm{supp}(V)) = \mathrm{supp}(WVW^{-1})$, we have $\mathrm{supp}(U_{1,2} T_2' U_{1,2}) \subseteq X_2$. Let $T_2 = U_{1,2} T_2' U_{1,2}$, so that $T_1 = S_1 U_{1,2} T_2 U_{1,2} = S_1 (U_{1,2} T_2 U_{1,2} T_2^{-1}) T_2 = S_1 S_{1,2} T_2$, where $S_{1,2}$ is a commutator of elements supported by $X_1 \cup X_2$. Continuing this way, we write $T_2 = S_2 S_{2,3} T_3$, where S_2 is a commutator of elements supported by X_2 and $S_{2,3}$ a commutator of elements supported in $X_2 \cup X_3$ and T_3 is supported by X_3, etc. Then if $\bar{S}_n = S_n S_{n,n+1}$, $\bar{S}_n$ is the product of two commutators supported by $X_n \cup X_{n+1}$, $T_1 = \bar{S}_1 T_2$, $T_2 = \bar{S}_2 T_3, \dots$, with T_n supported by X_n. Let P_1 be equal to $\bar{S}_{2n+1}, n \geq 0$, on $X_{2n+1} \cup X_{2n+2}$ and P_2 be equal to $\bar{S}_{2n+2}, n \geq 0$, on $X_{2n+2} \cup X_{2n+3}$ and the identity otherwise. Then

$$\begin{aligned} P_1 = \bar{S}_1 \bar{S}_3 \bar{S}_5 \dots &= (T_1 T_2^{-1})(T_3 T_4^{-1})(T_5 T_6^{-1}) \dots \\ &= (T_1 T_3 T_5 \dots)(T_2^{-1} T_4^{-1} T_6^{-1} \dots) \end{aligned}$$

and

$$\begin{aligned} P_2 = \bar{S}_2 \bar{S}_4 \bar{S}_6 \dots &= (T_2 T_3^{-1})(T_4 T_5^{-1})(T_6 T_7^{-1}) \dots \\ &= (T_2 T_4 T_6 \dots)(T_3^{-1} T_5^{-1} T_7^{-1} \dots). \end{aligned}$$

Thus $T_1 = P_1 P_2$ and P_1, P_2 are each a product of 2 commutators. □

Lemma 4.5. *Let E be ergodic. Then every element of $[E]$ is a product of 10 involutions in $[E]$.*

Proof. Let $T \in [E]$. All transformations below are in $[E]$. As in the preceding proof, we write $T = S_0 T_1$, where $\delta_u(T_1, 1) \leq \frac{1}{2}$ and, by 4.4, S_0 a product of 2 involutions. Then, proceeding as before, we have $T_1 = S_1 T_2' = S_1 (T_2' U_{1,2} (T_2')^{-1}) U_{1,2} (U_{1,2} T_2' U_{1,2})$, with S_1 a product of 2 involutions, and note that $U_{1,2}, T_2' U_{1,2} (T_2')^{-1}$ are involutions. So $T_1 = S_1 S_{1,2}' T_2$, where $S_{1,2}' = (T_2' U_{1,2} (T_2')^{-1}) U_{1,2}$ is the product of two involutions. Thus, continuing as in the rest of the proof, T_1 is the product of two transformations each of which is the product of 4 involutions and we are done. □

We now have as a consequence of 4.2 the following result.

Theorem 4.6. *If E is ergodic, then $[E]$ is a simple group.*

Proof. Let $N \lhd [E], N \neq \{1\}$. We will show that $N = [E]$. Notice that for any group G and $N \lhd G$, if H generates G and $[H, H] \subseteq N$, then also

$[G, G] \subseteq N$. Indeed if $\pi : G \to G/N$ is the canonical projection, every two elements of $\pi(H)$ commute, so since $\pi(H)$ generates $\pi(G), \pi(G)$ is abelian, so $[G, G] \subseteq N$. Also notice that for each $\epsilon > 0$, the elements $T \in [E]$ with $\delta_u(T, 1) < \epsilon$ generate $[E]$. This follows immediately from 4.5 and the fact that any involution in $[E]$ is a product of involutions in $[E]$ with arbitrarily small support (or alternatively one can use the path connectedness of $[E]$, see 3.12). Finally fix $T_0 \neq 1$ in N and then a set A of positive measure with $T_0(A) \cap A = \emptyset$. Let $\epsilon = \mu(A)/2$. We will show that the commutator of any two elements of $[E]$ with $\delta_u(T, 1) < \epsilon$ is in N.

Fix $S, T \in [E]$ with $\delta_u(T, 1), \delta_u(S, 1) < \epsilon$. Conjugating, if necessary, we can assume that their supports are contained in A. Thus $S^{-1}, T_0T^{-1}T_0^{-1}$ have disjoint supports, so commute. Consider $\hat{T} = (TT_0T^{-1})T_0^{-1} \in N$. Then

$$\begin{aligned} [S, T] &= STS^{-1}T^{-1} \\ &= ST(T_0T^{-1}T_0^{-1})S^{-1}(T_0TT_0^{-1}T^{-1}) \\ &= S(TT_0T^{-1}T_0^{-1})S^{-1}(T_0TT_0^{-1}T^{-1}) \\ &= S\hat{T}S^{-1}(\hat{T})^{-1} \in N. \end{aligned}$$

□

Note that conversely simplicity of $[E]$ implies that E is ergodic. Indeed, if A is an E-invariant Borel set which is neither null or conull, then the elements of $[E]$ supported by A form a non-trivial closed normal subgroup of $[E]$. In fact, Bezuglyi-Golodets [BG1] have shown that in case E is aperiodic, non-ergodic, all the nontrivial closed normal subgroups of $[E]$ arise this way by letting A vary over the invariant Borel sets.

We need one more lemma before embarking in the proof of 4.1.

Lemma 4.7. *Let E be ergodic. Let $1 \neq T \in [E]$ be an involution, C_T its centralizer in $[E]$. Then $[T]$ is the largest abelian normal subgroup of C_T.*

Proof. Note that for $S, T' \in \mathrm{Aut}(X, \mu), S[T']S^{-1} = [ST'S^{-1}]$. So if $S \in C_T, S[T]S^{-1} = [T]$, i.e., $[T] \lhd C_T$. Clearly, as T is an involution, $[T]$ is abelian.

Let now $N \lhd C_T$, N abelian. Consider the standard Borel space $X/T = X/E_T$, the projection $\pi : X \to X/T$ and the measure $\nu = \pi_*\mu$. Then E/T is a countable, measure preserving, ergodic equivalence relation on $(X/T, \nu)$. Clearly every $S \in C_T$ preserves the support of T and induces a map S^* in the full group of E/T restricted to $\mathrm{supp}(T)/T$; $\rho(S) = S^*$ is clearly a surjective homomorphism. So $\rho(N)$ is a normal subgroup of $[(E/T)|(\mathrm{supp}(T)/T)]$, which is simple, by 4.6. Since $\rho(N)$ is abelian, $\rho(N) \neq [(E/T)|(\mathrm{supp}(T)/T)]$, thus $\rho(N) = \{1\}$, i.e., $N \subseteq \ker(\rho) = C_T \cap [T] \subseteq [T]$, which completes the proof. □

(C) We are now ready to give the proof of 4.1. It is clearly enough to show (iv) ⇒ (i) and the "moreover" assertion.

So fix an algebraic isomorphism $f : [E] \to [F]$. It follows from 4.7 that if $T \in [E], 1 \neq T, T^2 = 1$, then $f([T]) = [f(T)]$.

Let for $T \in [E]$ (and similarly for $[F]$),

$$\perp(T) = \{S \in [E] : \mathrm{supp}(S) \cap \mathrm{supp}(T) = \emptyset\}.$$

Lemma 4.8. *If $T \neq 1$ is an involution, then $f(\perp(T)) = \perp(f(T))$.*

Proof. Note that $\perp(T)$ is really the same as $[E|(X \setminus \mathrm{supp}(T))]$, so every element of $\perp(T)$ is the product of commutators in $\perp(T)$, by 4.2. Thus the same holds for $f(\perp(T))$. We will show that $f(\perp(T)) \subseteq \perp(f(T))$. Applying this to $f^{-1}, f(T)$ we get $f^{-1}(\perp(f(T))) \subseteq \perp(T)$, so we have equality.

Claim. *If $S \in C_{[T]}$, where $C_{[T]}$ is the centralizer of $[T]$ in $[E]$, then $S = S_1 S_2$, where $S_1 \in [T], S_2 \in \perp(T)$.*

Granting this claim, we complete the proof of 4.8. Take any commutator in $f(\perp(T))$, say $[U_1, U_2]$, with $U_1, U_2 \in f(\perp(T))$. Then, as $\perp(T) \subseteq C_{[T]}$, $U_i \in f(C_{[T]}) = C_{[f(T)]}$ (this is because $[T]$ and thus $C_{[T]}$ is preserved under f, by 4.7). So, by the claim, applied to $f(T)$, $U_i = T_i V_i$ with $T_i \in [f(T)], V_i \in \perp(f(T))$. Thus

$$U_1 U_2 U_1^{-1} U_2^{-1} = T_1 V_1 T_2 V_2 V_1^{-1} T_1^{-1} V_2^{-1} T_2^{-1}$$

and, since T_i, V_j commute, this is equal to

$$V_1 V_2 V_1^{-1} V_2^{-1} \in \perp(f(T)).$$

Proof of the claim. If $S \in C_{[T]}$, then $S \in C_T$, so $S(\mathrm{supp}(T)) = \mathrm{supp}(T)$, thus $S = S_1 S_2$, where $S_1 = S|\mathrm{supp}(T) \cup \mathrm{id}|(X \setminus \mathrm{supp}(T)), S_2 = S|(X \setminus \mathrm{supp}(T)) \cup \mathrm{id}|\mathrm{supp}(T)$. Clearly $S_2 \in \perp(T)$, so it is enough to show that $S_1 \in [T]$. By restricting everything to the support of T, if necessary, it is enough to show that if $\mathrm{supp}(T) = X$ and $S \in C_{[T]}$, then $S \in [T]$. Now S descends to S^* on X/T, so if $S \notin [T], S^* \neq 1$ on X/T, thus there is a T-invariant set $A \subseteq X$ of positive measure with $S(A) \cap A = \emptyset$. Then $S(A)$ is also T-invariant. Define now $U \in [T]$ to be the identity on $X \setminus S(A)$ and switch any two elements of $S(A)$ in the same T-orbit. Then $US \neq SU$, a contradiction. □

Now take any $A \in \mathrm{MALG}_\mu$. Then there is an involution T in $[E]$ with $\mathrm{supp}(T) = A$ (split A into two sets of equal measure and let T map one onto the other and be the identity off A). Put

$$\Phi(A) = \mathrm{supp}(f(T)) \in \mathrm{MALG}_\mu.$$

Claim. *This is well-defined.*

Proof of the claim. Indeed, if T' is another involution with $\mathrm{supp}(T') = A$, we have $\perp(T) = \perp(T')$, so $\perp(f(T)) = \perp(f(T'))$, by 4.8. Note now that for any $U, V \in [F], \mathrm{supp}(U) \subseteq \mathrm{supp}(V) \Leftrightarrow \perp(V) \subseteq \perp(U)$, so $\mathrm{supp}(U) = \mathrm{supp}(V) \Leftrightarrow \perp(V) = \perp(U)$, thus $\mathrm{supp}(f(T)) = \mathrm{supp}(f(T'))$. □

Clearly Φ is a bijection of MALG_μ with itself with inverse

$$\Phi^{-1}(B) = \mathrm{supp}(f^{-1}(T')),$$

where T' is an involution with support B. Also if $\mathrm{supp}(T) = A$, $\mathrm{supp}(T') = B$,

$$\begin{aligned} A \subseteq B \Leftrightarrow\ & \mathrm{supp}(T) \subseteq \mathrm{supp}(T') \\ \Leftrightarrow\ & \perp(T') \subseteq \perp(T) \\ \Leftrightarrow\ & \perp(f(T')) \subseteq \perp(f(T)) \\ \Leftrightarrow\ & \mathrm{supp}(f(T)) \subseteq \mathrm{supp}(f(T')) \\ \Leftrightarrow\ & \Phi(A) \subseteq \Phi(B). \end{aligned}$$

Thus Φ is an automorphism of the *Boolean* algebra MALG_μ (not necessarily the *measure* algebra MALG_μ), so it is given by a (unique modulo null sets) Borel automorphism $\varphi : X \to X$ which is non-singular, i.e., preserves null sets, via

$$\Phi(A) = \varphi(A).$$

We now want to verify that actually φ is measure preserving, i.e., $\varphi \in \mathrm{Aut}(X, \mu)$, and moreover that $f(S) = \varphi S \varphi^{-1}, \forall S \in [E]$.

Lemma 4.9. *For each $S \in [E], f(S) = \varphi S \varphi^{-1}$.*

Granting this we verify that $\varphi \in \mathrm{Aut}(X, \mu)$. Since φ is non-singular, there is measurable $r : X \to \mathbb{R}^+$ with $\mu(\varphi(A)) = \int_A r d\mu, \forall A \in \mathrm{MALG}_\mu$. We will show that r is actually constant, $r = k$. Then $\mu(\varphi(A)) = k\mu(A)$, so, putting $A = X, k = 1$, thus $\varphi \in \mathrm{Aut}(X, \mu)$.

If r is not constant, there are two disjoint sets of positive measure A, B with $0 \leq r(x) < \lambda < r(y)$, for $x \in A, y \in B$. Let $C \subseteq A, D \subseteq B$ be such that $\mu(C) = \mu(D) > 0$ and let $K = \varphi(C)$. Let also $U \in [E]$ be such that $U(C) = D$. Then $\mu(\varphi(D)) = \int_D r d\mu > \lambda\mu(D) = \lambda\mu(C), \mu(K) = \mu(\varphi(C)) = \int_C r d\mu < \lambda\mu(C) < \mu(\varphi(D))$. So $\mu(K) < \mu(\varphi(D))$ and $\varphi(D) = \varphi U \varphi^{-1}(K)$, so $\mu(K) < \mu(\varphi U \varphi^{-1}(K))$, thus $\varphi U \varphi^{-1} \notin \mathrm{Aut}(X, \mu)$, contradicting that $\varphi U \varphi^{-1} = f(U) \in [F] \subseteq \mathrm{Aut}(X, \mu)$.

Proof of Lemma 4.9. As $[E]$ is generated by involutions, it is enough to prove this for $S = T$, an involution. Notice that if $T \in [E]$ is an involution, then

$$\mathrm{supp}(\varphi T \varphi^{-1}) = \varphi(\mathrm{supp}(T)) = \Phi(\mathrm{supp}(T)) = \mathrm{supp}(f(T)),$$

i.e., $\varphi T \varphi^{-1}$ and $f(T)$ have the same support A. Assume, towards a contradiction, that

$$B = \{x \in A : \varphi T \varphi^{-1}(x) \neq f(T)(x)\}$$

has positive measure. Then we can find $C \subseteq B$ of positive measure such that

$$\begin{aligned} \varphi T\varphi^{-1}(C) \cap C &= \emptyset, \\ \varphi T\varphi^{-1}(C) \cap f(T)(C) &= \emptyset, \\ f(T)(C) \cap C &= \emptyset. \end{aligned}$$

Let $D = C \cup f(T)(C)$. Then D is $f(T)$-invariant but not $\varphi T\varphi^{-1}$ invariant. Write $f(T) = U_1U_2$, where $U_1 = f(T)$ on $D, =$ id on $X \setminus D$ and $U_2 = f(T)$ on $X \setminus D$, id on D, so that $U_1, U_2 \in [F]$ are involutions. Then

$$T = f^{-1}(U_1)f^{-1}(U_2),$$

so $\varphi T\varphi^{-1} = (\varphi f^{-1}(U_1)\varphi^{-1})(\varphi f^{-1}(U_2)\varphi^{-1})$. Now $\varphi f^{-1}(U_1)\varphi^{-1}$ has support $\varphi(\mathrm{supp}(f^{-1}(U_1))) = \mathrm{supp}(ff^{-1}(U_1)) = \mathrm{supp}(U_1)$ and similarly the automorphism $\varphi f^{-1}(U_2)\varphi^{-1}$ has support equal to $\mathrm{supp}(U_2)$, thus $\varphi T\varphi^{-1}$ leaves D invariant, a contradiction. □

Finally, we verify that there is unique φ such that $f(S) = \varphi S\varphi^{-1}$. For that we use the following lemmas which, for further reference, we state in more generality than we need here.

Lemma 4.10. *For any aperiodic E (not necessarily ergodic) and Borel set A, there is $T \in [E]$ with* $\mathrm{supp}(T) = A$.

Proof. Since E is aperiodic, measure preserving, we can assume that $E|A$ is aperiodic. Thus, by 3.5, there is aperiodic $T_0 : A \to A$ with $T_0(x)Ex$. Let $T = T_0 \cup \mathrm{id}|(X \setminus A)$. □

Corollary 4.11. *For any aperiodic E (not necessarily ergodic) the centralizer $C_{[E]}$ of $[E]$ in* $\mathrm{Aut}(X, \mu)$ *is trivial.*

Proof. If $S \in C_{[E]}$ and $A \in \mathrm{MALG}_\mu$, then by 4.10 there is $T \in [E]$ with $\mathrm{supp}(T) = A$. Then $A = \mathrm{supp}(T) = \mathrm{supp}(STS^{-1}) = S(A)$, so $S = 1$. □

The uniqueness now follows immediately from 4.11 and the proof of 4.1 is complete.

Remark. Note that the argument following the statement of 4.9 shows that if φ is a non-singular Borel automorphism of X, E is ergodic and $\varphi[E]\varphi^{-1} \subseteq \mathrm{Aut}(X, \mu)$, then $\varphi \in \mathrm{Aut}(X, \mu)$.

(D) Dye's Reconstruction Theorem suggests the problem of distinguishing, up to isomorphism, ergodic equivalence relations by finding algebraic or perhaps topological group distinctions of their corresponding full groups (it is understood here that full groups are equipped with the uniform topology). For example, 3.17 provides such a distinction between hyperfinite and non-hyperfinite ones.

One might explore the following possibility. Recall that an action of Γ on (X, μ) is called a *free action* if $\forall \gamma \neq 1(\gamma \cdot x \neq x,\ \mu\text{-a.e.})$. We know by Gaboriau [Ga1] that if E_m is given by a free, measure preserving action of the free group F_m and E_n by a similar action of the free group F_n ($1 \leq m, n \leq \infty$), then if $m \neq n, E_m \not\cong E_n$. Can one detect this by looking at their full groups $[E_m], [E_n]$? One can look for example at topological generators. Recall that given a topological group G a subset of G is a *topological generator* if it generates a dense subgroup of G. Denote by $t(G)$ the smallest cardinality of a set of topological generators of G. (Thus $t(G) = 1$ iff G is monothetic.) Clearly $1 \leq t(G) \leq \aleph_0$, if G is separable.

In the notation above, and keeping in mind that if Γ is dense in $([E], u)$, then $E = E_\Gamma^X$, Gaboriau's [Ga1] theory of costs immediately implies that $t([E_n]) \geq n$. In fact, Ben Miller pointed out that one actually has $t([E_n]) \geq n + 1$. To see this, assume, towards a contradiction, that $T_1, \dots, T_n$ are topological generators for $[E_n]$ and let Γ be the group they generate. Since $E_n = E_\Gamma^X$, and E_n has cost n, it follows from Gaboriau [Ga1], I.11, that Γ acts freely. Let then S_1, S_2 be two distinct elements of Γ and (as in the proof of 3.10) find disjoint Borel sets A_1, A_2 of positive measure such that $S_1(A_1), S_2(A_2)$ are also disjoint. Then there is $S \in [E_n]$ with $S|A_i = S_i|A_i$, for $i = 1, 2$. Clearly, S cannot be in the uniform closure of Γ.

In particular, $t([E_\infty]) = \aleph_0$. Note that, by 3.9, for any ergodic E we have $t([E]) \geq 2$. In an earlier version of this work, I have raised the question of whether $t([E_n]) < \infty$ (even in the case $n = 1$). This has now been answered by the following result.

Theorem 4.12 (Kittrell-Tsankov [KiT]). *An ergodic equivalence relation E is generated by an action of a finitely generated group iff $t([E]) < \infty$ (where $[E]$ is equipped with the uniform topology). Moreover if E_n is generated by a free, measure preserving, ergodic action of F_n, then $t([E_n]) \leq 3(n + 1)$.*

Kittrell and Tsankov [KiT] also proved that if $n = 1$, i.e., for ergodic *hyperfinite* E, we have $t([E]) \leq 3$. It is unknown whether in this case the value of $t([E])$ is 2. For any $n < \infty$, we have $n + 1 \leq t([E_n]) \leq 3(n + 1)$. It is unknown if $t([E_n])$ is independent of the action and, in case this has a positive answer, what is the exact value of $t([E_n])$. In any case, the preceding shows that $t([E_n]) < t([E_m])$, provided $m+1 > 3(n+1)$. This appears to be the first result providing a topological group distinction between $[E_m], [E_n]$, provided m, n are sufficiently far apart.

We remark that it is known that $t(\mathrm{Aut}(X, \mu), w) = 2$ and in fact the set of pairs $(g, h) \in \mathrm{Aut}(X, \mu)^2$ that generate a dense subgroup is dense G_δ in $\mathrm{Aut}(X, \mu)^2$ (see Grzaslewicz [Gr], Prasad [Pr], and Kechris-Rosendal [KR] for a different approach). Also it is not hard to see that if $U(H)$ is the unitary group of an infinite dimensional Hilbert space, then $t(U(H)) = 2$, and again, in fact, the set of pairs $(g, h) \in U(H)^2$ that generate a dense subgroup of $U(H)$ is dense G_δ in $U(H)^2$. This is because if $U(n)$ is the unitary group of the finite-dimensional Hilbert space $\mathbb{C}^n$ then (with some canonical identifications) $U(1) \subseteq U(2) \subseteq \cdots \subseteq U(H)$ and $\overline{\bigcup_n U(n)} = U(H)$. Then since each $U(n)$ is compact and connected, it follows, e.g., by a result of Schreier-Ulam [SU], that the set of pairs $(g,h) \in U(n)^2$ that generate a dense subgroup of $U(n)$ is dense G_δ in $U(n)^2$. The same result for $U(H)$ follows, using the Baire Category Theorem, since for each $u \in \bigcup_n U(n)$ and open nbhd N of u, the set of $(g,h) \in U(H)^2$ that generate a subgroup intersecting N is open dense.

It is perhaps worth pointing out here that although, as we have seen above, $([E],u)$ is topologically finitely generated, when E is given by an ergodic action of a finitely generated group, for any E and for any $n \geq 1$ it is *not* the case that the set of n-tuples $(T_1,\dots,T_n) \in [E]^n$ that generate a dense subgroup of $([E],u)$ is dense in $[E]^n$ (with the product uniform topology). To see this, simply notice that if $T_1,\dots,T_n$ satisfy $d_u(T_i,1) < \epsilon$ for $i = 1,\dots,n$, then there is a set $A \subseteq X$ with $\mu(A) > 1 - n\epsilon$ such that $T_i|A = \mathrm{id}$, $\forall i \leq n$, so $T|A = \mathrm{id}$ for each $T \in \langle T_1,\dots,T_n\rangle$, thus if $n\epsilon < 1, \langle T_1,\dots,T_n\rangle \subseteq \{T \in [E] : d_u(T,1) < n\epsilon\}$ is not dense in $([E],u)$.

This simple argument also limits the kinds of Polish groups that can be closed subgroups of $([E],u)$ for an equivalence relation E. Call a Polish group G *locally topologically finitely generated* if there is $n \geq 1$ such that for any open nbhd V of $1 \in G$, there are $g_1,\dots,g_n \in G$ with $\langle g_1,\dots,g_n\rangle$ dense in G. Examples of such groups include $\mathbb{R}^n, \mathbb{T}^n, U(H), \mathrm{Aut}(X,\mu)$. Also factors and finite products of locally topologically finitely generated groups have the same property. Now notice that by the above argument any continuous homomorphism of such a group into the full group $([E],u)$ must be trivial. In particular, a non-trivial locally topologically finitely generated group cannot be a closed subgroup of $([E],u)$ (or even continuously embed into $([E],u)$).

(E) Recently Pestov (private communication) raised the following related question: Let E be a measure preserving, ergodic, hyperfinite equivalence relation. What kind of countable groups embed (algebraically) into $[E]$? In response to this we mention the following two facts. For the first, recall that a countable group Γ is *residually finite* (resp., *residually amenable*) if for any $\gamma \in \Gamma, \gamma \neq 1$ there is an homomorphism $\pi : \Gamma \to \Delta$, where Δ is finite (resp., amenable) such that $\gamma \notin \ker(\pi)$. The following simple result was originally proved for residually finite groups. Ben Miller then noticed that the argument really shows the following stronger fact.

Proposition 4.13. *Let E be an ergodic, hyperfinite equivalence relation. Given a countable group Γ, if for every $\gamma \in \Gamma \setminus \{1\}$ there is a homomorphism $\pi : \Gamma \to [E]$ such that $\gamma \notin \ker(\pi)$, then Γ embeds into $[E]$. In particular every residually amenable Γ embeds into $[E]$ and for every countable group Γ there is unique normal subgroup N such that Γ/N embeds into $[E]$ and every homomorphism from N into $[E]$ is trivial.*

Proof. Suppose (X,μ) be the space on which E lives. Let $\Gamma \setminus \{1\} = \{\gamma_1,\gamma_2,\dots\}$. Fix then a sequence of homomorphisms $\pi_n : \Gamma \to [E]$ such that $\gamma_n \notin \ker(\pi_n)$. Next split X into countably many Borel sets $A_1, A_2,\dots$, each meeting every E-class. Let $E_n = E|A_n$. Considering the space A_n with the normalized restriction of μ to A_n, E_n is hyperfinite and ergodic, so, by

3.13, it is isomorphic to E, and thus, using π_n, there is a Borel measure preserving action a_n of Γ on A_n such that γ_n acts non-trivially and the equivalence relation induced by a_n is contained in E_n. Taking the union of these actions on the A_n's gives an action on Γ on X such that every $\gamma \neq 1$ acts non-trivially and the equivalence relation induced by this action is included in E. This clearly gives an embedding of Γ into $[E]$.

Since every amenable group can be clearly embedded into $[E]$, by the result of Ornstein-Weiss [OW] and 3.13, it is clear that every residually amenable Γ embeds into $[E]$. Finally, the last assertion of the proposition is clear by taking N to be the intersection of the kernels of all homomorphisms from Γ into $[E]$. □

We also have the following partial converse. It is a special case of a result of Robertson [Ro], concerning groups embeddable into the unitary group of the hyperfinite II_1 factor but we give below a direct ergodic theory argument.

Below we recall that a Borel (not necessarily countable) equivalence relation R on a standard Borel space Y is *smooth* if there is Borel map $f : Y \to Z$, Z a standard Borel space, such that $xRy \Leftrightarrow f(x) = f(y)$. If R is countable, this is equivalent to the existence of a Borel selector for R. Finally, if E is countable, measure preserving on (X, μ), then E is called smooth if its restriction to a co-null E-invariant Borel set is smooth in the previous sense. It is then easy to see that E is smooth iff E is finite.

Proposition 4.14. *Let E be an ergodic, hyperfinite equivalence relation. If Γ is a countable group, $\Gamma \leq [E]$ and Γ has property* (T)*, then Γ is residually finite.*

Proof. Let $F \subseteq E$ be the equivalence relation induced by Γ.

Claim. *F is smooth.*

Granting this, we can complete the proof as follows. Since F is smooth and measure preserving it must have finite classes a.e. Decompose then X into countably many Γ-invariant Borel sets $A_1, A_2, \dots$ such that the Γ-orbits in A_n have cardinality n. Fixing a Borel linear ordering on X, we can identify each Γ-orbit in A_n with $\{1, 2, \dots, n\}$ and thus each Γ-orbit in A_n gives rise to a homomorphism of Γ into S_n. Let $\gamma \in \Gamma$ be different from 1. Then for some n, γ acts non-trivially on A_n, so the homomorphism corresponding to some Γ-orbit in A_n sends γ to something different from 1 in S_n. So Γ is residually finite.

Proof of the claim. We will use the following result of Schmidt-Zimmer (see Zimmer [Zi1], 9.1.1): If a property (T) group Γ acts in an ergodic, measure preserving way on a standard measure space (Y, ν) and $\alpha : \Gamma \times Y \to \mathbb{Z}$ is a Borel cocycle of this action, then α is a coboundary, i.e., $\alpha(\gamma, x) = f(\gamma \cdot x) - f(x)$, for some Borel $f : X \to \mathbb{Z}$.

Fix a free Borel action $(n, x) \mapsto n \cdot x$ of $\mathbb{Z}$ on (X, μ) which generates E. To show that F is smooth, it is enough to show that if (Y, ν) is one of the ergodic components of the action of Γ, with ν non-atomic, then $F|Y$ is smooth. Consider the cocycle α from $\Gamma \times Y$ into $\mathbb{Z} : \alpha(\gamma, x) =$ (the unique n such that $\gamma \cdot x = n \cdot x$). Then there is Borel $f : Y \to \mathbb{Z}$ such that $\alpha(\gamma, x) = f(\gamma \cdot x) - f(x)$. Fix a set A of positive ν-measure, thus meeting every F-class in Y, on which f is constant. Then if $x, \gamma \cdot x \in A$ we have $\alpha(\gamma, x) = 0$, so $\gamma \cdot x = x$, i.e., A meets every F-class in exactly one point, so $F|Y$ is smooth. □

There are related questions that came up in a discussion with Sorin Popa. What countable groups Γ embed (algebraically) into $[E]$, for an ergodic, hyperfinite E, so that they generate E? More generally, what countable groups Γ have measure preserving, ergodic actions that generate a hyperfinite equivalence relation? Every group that has this last property does not have property (T) and one may wonder if conversely every non-property (T) group has such an action. This however fails. Fix a countable group Γ which is simple, not residually finite, and has property (T) (such groups exist as pointed out by Simon Thomas). Let F be a non-trivial finite group and let $\Delta = \Gamma * F$. Then Δ does not have property (T). Assume, towards a contradiction, that Δ has an ergodic action which generates a hyperfinite equivalence relation E. Then there is a homomorphism $\pi : \Delta \to [E]$. Clearly π is 1-1 or trivial on Γ. In the second case, $\pi(\Delta) = \pi(F)$ is a finite group generating E, which is absurd. In the first case Γ embeds into $[E]$, which contradicts 4.14.

On the other hand, one can see that the free groups embed into $[E]$, E ergodic, hyperfinite, so that they generate E. Take, for example, F_2. It is enough to find $S, T \in [E]$, which generate a free group that acts ergodically. But 3.9 shows that in the uniform topology the set of pairs (S, T) in $(\mathrm{APER} \cap [E])^2$ that generate a free group is dense G_δ and, using 3.6, it is easy to check that the set of such pairs that generate a group acting ergodically is also dense G_δ. Thus the generic pair in $(\mathrm{APER} \cap [E])^2$ works.

In general, it is not clear what countable groups embed into the full group of a given ergodic equivalence relation, not necessarily hyperfinite. A result of Ozawa [O] implies that there is no full group $[F]$ into which *every* countable group embeds (algebraically). In particular, there is no full group $[F]$ that contains up to conjugacy all full groups or, equivalently, there is no equivalence relation F such that for any other equivalence relation E there is $E' \subseteq F$ with $E \cong E'$. We give an ergodic theory proof of this fact in Section 14, **(C)**.

Ben Miller has pointed out that in the purely Borel theoretic context one actually has the opposite answer. Below all spaces are uncountable standard Borel spaces. If E, F are countable Borel equivalence relations on X, Y, $E \cong_B F$ means that there is a Borel bijection $\pi : X \to Y$ with $xEy \Leftrightarrow \pi(x)F\pi(y)$. Also $E \sqsubseteq_B F$ means that there is a Borel injection with $xEy \Leftrightarrow \pi(x)F\pi(y)$. Miller shows that there is a countable Borel equivalence

F such that for every countable Borel equivalence relation E there is $E' \subseteq F$ with $E \cong_B E'$. His argument goes as follows.

Let E_∞ be a countable Borel equivalence relation on X such that for any countable Borel equivalence relation E we have $E \sqsubseteq_B E_\infty$ (see Dougherty-Jackson-Kechris [DJK], 1.8). Let $I(\mathbb{N}) = \mathbb{N} \times \mathbb{N}$ and put $F = E_\infty \times I(\mathbb{N})$, an equivalence relation on $Y = X \times \mathbb{N}$. We will show that F works. Take first a non-smooth E (on some space Z). Let $\pi : Z \to X$ be Borel and 1-1 that witnesses $E \sqsubseteq_B E_\infty$. Put

$$\pi'(z) = (\pi(z), 0).$$

Then π' witnesses $E \sqsubseteq_B F$. Let $P = \pi'(Z)$. Clearly $F|(Y \setminus P)$ is compressible, via $(x, n) \mapsto (x, n+1)$, so there is an aperiodic smooth subequivalence relation $R \subseteq F|(Y \setminus P)$ (see [DJK], 2.5). Let $E' = (F|P) \cup R$. Clearly $E' \subseteq F$, so it is enough to show that $E' \cong_B E$. Now $E' \cong_B E \oplus R$ and clearly there is an E-invariant Borel set $A \subseteq Z$ such that $E|A \cong_B R \oplus R \oplus \dots$ (we are using here that E is not smooth). So

$$\begin{aligned} E' \cong_B E \oplus R &\cong_B (E|(Z \setminus A) \oplus R \oplus \dots) \oplus R \\ &\cong_B (E|(Z \setminus A)) \oplus (R \oplus R \oplus \dots) \cong_B E. \end{aligned}$$

Finally, when E is smooth, the proof is easy as F contains an aperiodic smooth subequivalence relation.

Comments. The method of proof and most of the results about $[E]$ up to 4.11 come (with some modifications) from the papers Eigen [Ei1, Ei2] and Fathi [Fa]. The numbers 5 and 10 in 4.2 and 4.5 can be replaced by 1, 3 resp., see Miller [Mi].

5. Turbulence of conjugacy

(A) Foreman and Weiss [FW] have shown that the conjugacy action of the group $(\mathrm{Aut}(X, \mu), w)$ on (ERG, w) is *turbulent*. We will provide below (see 5.3) a different proof of (a somewhat stronger version of) this result. Our argument also shows that the conjugacy action of $([E], u)$ on $(\mathrm{APER} \cap [E], u)$ is turbulent for any ergodic but not E_0-ergodic E. We will first give the proof of this for hyperfinite E in order to explain in a somewhat simpler context the main idea. In this section, we use Hjorth [Hj2] and Kechris [Kec3] as references for the basic concepts and results of Hjorth's theory of turbulence.

Theorem 5.1 (A special case of 5.2). *Let E be a hyperfinite ergodic equivalence relation. Then the conjugacy action of $([E], u)$ on $(\mathrm{APER} \cap [E], u)$ is turbulent.*

Proof. We have already seen in 3.4 that if $T \in \mathrm{APER} \cap [E]$, then its conjugacy class (in $[E]$) is dense in $(\mathrm{APER} \cap [E], u)$. We next verify that if $T \in \mathrm{APER} \cap [E]$, then its conjugacy class (in $[E]$) is meager in $(\mathrm{APER} \cap [E], u)$. First, by 2.5, the conjugacy class of T in $(\mathrm{Aut}(X, \mu), w)$ is meager, so it is disjoint from a conjugacy-invariant dense G_δ in $(\mathrm{Aut}(X, \mu), w)$, say C. We can also assume that $C \subseteq \mathrm{APER}$ by 2.3. By 3.14, $D = C \cap [E] \neq \emptyset$.

Clearly $D \subseteq \mathrm{APER} \cap [E]$ is conjugacy invariant in $[E]$ and, being nonempty, it is dense in $(\mathrm{APER} \cap [E], u)$ by 3.4. Finally as $w \subseteq u, C$ is G_δ in $(\mathrm{Aut}(X, \mu), u)$ and so D is G_δ in $(\mathrm{APER} \cap [E], u)$. As D is disjoint from the conjugacy class of T in $[E]$, we are done.

We finally verify that if $T \in \mathrm{APER} \cap [E]$, then T is turbulent (for the conjugacy action of $([E], u)$ on $(\mathrm{APER} \cap [E], u)$).

Fix $\epsilon > 0$ and let

$$U_\epsilon = \{S \in \mathrm{APER} \cap [E] : \delta'_u(S, T) < \epsilon\}.$$

Fix also a nbhd V of 1 in $([E], u)$. It is enough to show that the local orbit

$$\mathcal{O}(T, U_\epsilon, V)$$

is dense in U_ϵ.

Since $\{gTg^{-1} : g \in [E]\}$ is dense in $(\mathrm{APER} \cap [E], u)$, it is clearly dense in (U_ϵ, u). So fix any $g \in [E]$ with $gTg^{-1} \in U_\epsilon$. It is enough to show that $gTg^{-1} \in \overline{\mathcal{O}(T, U_\epsilon, V)}^u$.

Let $E = \bigcup_{n=1}^\infty E_n$, where $E_1 \subseteq E_2 \subseteq \dots$ are Borel with $\mathrm{card}([x]_{E_n}) \leq n, \forall x$. Then $\bigcup_n [E_n]$ is dense in $([E], u)$. Indeed, given $S \in [E]$, if $X_n = \{x : S(x) \in [x]_{E_n}\}$, then $X_1 \subseteq X_2 \subseteq \dots$ and $\bigcup_n X_n = X$. So for any $\rho > 0$, choose n large enough so that $\mu(X_n) > 1 - \rho$. Then, as $x \in X_n \Rightarrow S(x) \in [x]_{E_n}$, we can find $S' \in [E_n]$ such that $S|X_n = S'|X_n$ and so $\delta_u(S, S') < \rho$.

So we can clearly assume that $g \in [E_n]$, for some large enough n.

Notice that it is enough to find a continuous path $\lambda \mapsto g_\lambda$ in $([E], u), \lambda \in [0, 1]$, such that $g_0 = 1, g_1 = g$ and $g_\lambda T g_\lambda^{-1} \in U_\epsilon, \forall \lambda$. Because then we can find $\lambda_0 = 0 < \lambda_1 < \dots < \lambda_k = 1$ with $g_{\lambda_{i+1}} g_{\lambda_i}^{-1} \in V, \forall i < k$. If $T_1 = g_{\lambda_1}, T_2 = g_{\lambda_2} g_{\lambda_1}^{-1}, \dots, T_k = g_{\lambda_k} g_{\lambda_{k-1}}^{-1}$, then $T_i \in V$ and

$$T_i T_{i-1} \dots T_1 T T_1^{-1} \dots T_{i-1}^{-1} T_i^{-1} = g_{\lambda_i} T g_{\lambda_i}^{-1} \in U_\epsilon, \forall i \leq k.$$

Choose δ_0 small enough so that if $\alpha = \delta'_u(gTg^{-1}, T)$, then $\alpha + 4\delta_0 < \epsilon$. Find then $N \geq n$ large enough, so that there is $S \in [E_N]$ with $\delta'_u(S, T) < \delta_0$. Then $\delta'_u(S, gSg^{-1}) < \delta_0 + \delta'_u(T, gTg^{-1}) + \delta_0 = \alpha + 2\delta_0$.

Claim. *There is a continuous path $\lambda \mapsto g_\lambda$ in $([E], u)$ with $g_0 = 1, g_1 = g$ and $\delta'_u(S, g_\lambda S g_\lambda^{-1}) < \alpha + 2\delta_0$.*

Then $\delta'_u(T, g_\lambda T g_\lambda^{-1}) < \delta_0 + \alpha + 2\delta_0 + \delta_0 = \alpha + 4\delta_0 < \epsilon$, i.e., $g_\lambda T g_\lambda^{-1} \in U_\epsilon, \forall \lambda$.

Proof of the claim. Let Z be a Borel transversal for E_N and let $\lambda \mapsto Z_\lambda$ be continuous from $[0, 1]$ into MALG_μ, so that $Z_0 = \emptyset \subseteq Z_\lambda \subseteq Z_\nu \subseteq Z_1 = Z, \mu(Z_\nu \setminus Z_\lambda) \leq (\nu - \lambda)$, for $0 \leq \lambda \leq \nu \leq 1$. Let $X_\lambda = [Z_\lambda]_{E_N}$, so that $\mu(X_\nu \setminus X_\lambda) \leq N \cdot (\nu - \lambda)$. Finally, define

$$g_\lambda = \begin{cases} g \text{ on } X_\lambda, \\ \mathrm{id} \text{ on } X \setminus X_\lambda. \end{cases}$$

Then $\lambda \mapsto g_\lambda$ is a continuous path in $([E], u)$ and $g_0 = 1, g_1 = g$. Moreover, X_λ is g_λ, S-invariant for each λ and $g_\lambda | \bar{X}_\lambda = \mathrm{id} | \bar{X}_\lambda$, where $\bar{X}_\lambda = X \setminus X_\lambda$.

We check that $\delta'_u(S, g_\lambda S g_\lambda^{-1}) < \alpha + 2\delta_0$. Fix $A \in \mathrm{MALG}_\mu$. Then

$$\begin{aligned} g_\lambda S g_\lambda^{-1}(A) \Delta S(A) &= [g_\lambda S g_\lambda^{-1}(A \cap X_\lambda) \cup g_\lambda S g_\lambda^{-1}(A \cap \bar{X}_\lambda)] \Delta \\ &\quad [S(A \cap X_\lambda) \cup S(A \cap \bar{X}_\lambda)] \\ &= [g_\lambda S g_\lambda^{-1}(A \cap X_\lambda) \cup S(A \cap \bar{X}_\lambda)] \Delta \\ &\quad [S(A \cap X_\lambda) \cup S(A \cap \bar{X}_\lambda)] \\ &\subseteq g_\lambda S g_\lambda^{-1}(A \cap X_\lambda) \Delta S(A \cap X_\lambda) \\ &= [g S g^{-1}(A) \Delta S(A)] \cap X_\lambda, \end{aligned}$$

so $\mu(g_\lambda S g_\lambda^{-1}(A) \Delta S(A)) \leq \mu(g S g^{-1}(A) \Delta S(A)) \leq \delta'_u(g S g^{-1}, S) < \alpha + 2\delta_0$ and we are done. □

Remark. I would like to thank Greg Hjorth for suggesting a modification that simplified considerably my original calculation in the preceding proof.

We next show that a variation of the proof of 5.1 allows one to show the following stronger statement. Recall that an equivalence relation E is E_0-*ergodic* (or *strongly ergodic*) if for every *Borel homomorphism* $\pi : E \to E_0$, (i.e., a Borel map $\pi : X \to 2^{\mathbb{N}}$ with $xEy \Rightarrow \pi(x) E_0 \pi(y)$) the preimage of some E_0-class is conull. If $E = E_\Gamma^X$ for a Borel action of a countable group Γ on X and E is E_0-ergodic, we will also call this action E_0-*ergodic*. See Jones-Schmidt [JS] and Hjorth-Kechris [HK3], Appendix A, for some basic properties of this concept.

Theorem 5.2 (Kechris). *Let E be an ergodic equivalence relation which is not E_0-ergodic. Then the conjugacy action of* $([E], u)$ *on* $(\mathrm{APER} \cap [E], u)$ *is turbulent.*

Proof. Since E is not E_0-ergodic, there is a Borel homomorphism $\pi : E \to E_0$ such that the preimage of every E_0-class is μ-null. Write $E_0 = \bigcup_{n=1}^\infty E_n, E_1 \subseteq E_2 \subseteq \dots, E_n$ Borel with finite classes. We claim that the set $\bigcup_n [\pi^{-1}(E_n) \cap E]$ is dense in $([E], u)$. Indeed, if PER denotes the set of periodic elements of $\mathrm{Aut}(X, \mu)$, i.e.,

$$S \in \mathrm{PER} \Leftrightarrow \forall x \exists n (T^n(x) = x),$$

then, by 3.3, $\mathrm{PER} \cap [E]$ is dense in $([E], u)$. So fix a periodic $S \in [E]$ and let

$$X_n = \{x : \pi([x]_S) \text{ is contained in a single } E_n\text{-class}\},$$

where $[x]_S$ is the S-orbit of x. Then $X_1 \subseteq X_2 \subseteq \dots$ and $X = \bigcup_n X_n$. Also each X_n is S-invariant and if $S_n = S | X_n \cup \mathrm{id} | (X \setminus X_n)$, then $S_n \in [\pi^{-1}(E_n) \cap E]$ and $S_n \to S$ uniformly.

As in the proof of 5.1, whose notation we keep below, fix $T \in \mathrm{APER} \cap [E], U_\epsilon$, and $g \in [\pi^{-1}(E_n) \cap E]$ with $g T g^{-1} \in U_\epsilon$. Then choose δ_0 small enough so that $\alpha + 4\delta_0 < \epsilon$ and $N \geq n$ large enough so that for some $S \in [\pi^{-1}(E_N) \cap E], \delta'_u(S, T) < \delta_0$. It is again enough to find a continuous path $\lambda \mapsto g_\lambda$ with $g_0 = 1, g_1 = g$ and $\delta'_u(S, g_\lambda S g_\lambda^{-1}) < \alpha + 2\delta_0$.

Fix a Borel transversal A for E_N and let $\nu = \pi_*\mu$. Then $\nu(C) = 0$ for each E_N-class C. Define the measure ρ on A by

$$\rho(B) = \nu([B]_{E_N}), B \subseteq A \text{ Borel}.$$

Clearly it is non-atomic, so there is a continuous map $\lambda \mapsto A_\lambda$ from $[0,1]$ to MALG_ρ with $A_0 = \emptyset, A_1 = A$ and $0 \leq \lambda \leq \lambda' \leq 1 \Rightarrow A_\lambda \subseteq A_{\lambda'}$ and $\rho(A_{\lambda'} \setminus A_\lambda) \leq \lambda' - \lambda$. Let $X_\lambda = \pi^{-1}([A_\lambda]_{E_N})$, so that $\mu(X_\lambda) = \rho(A_\lambda)$, thus $\lambda \mapsto X_\lambda$ is continuous from $[0,1]$ to $\text{MALG}_\mu, X_0 = \emptyset, X_1 = X, 0 \leq \lambda \leq \lambda' \leq 1 \Rightarrow X_\lambda \subseteq X_{\lambda'}$ and $\mu(X_{\lambda'} \setminus X_\lambda) \leq \lambda' - \lambda$. Moreover X_λ is $\pi^{-1}(E_N)$-invariant, so g, S-invariant. Then define as before $g_\lambda = g|X_\lambda \cup \text{id}|(X \setminus X_\lambda)$, so that X_λ is also g_λ, S-invariant. The proof then proceeds exactly as in 5.1. □

With essentially the same proof one can show the result of Foreman-Weiss that the conjugacy action of $(\text{Aut}(X,\mu), w)$ on (ERG, w) is turbulent. Actually one can prove a somewhat more precise version (replacing ERG by APER) by using an additional fact (Lemma 5.4 below).

Theorem 5.3 (Foreman-Weiss [FW] for ERG). *The conjugacy action of the group* $(\text{Aut}(X,\mu), w)$ *on* (APER, w) *is turbulent.*

Proof. We have already seen that every conjugacy class in APER is dense and meager. We finally fix $T \in \text{APER}$ in order to show that it is turbulent for the conjugacy action. The following lemma was proved together with Ben Miller.

Lemma 5.4. *Every aperiodic, hyperfinite equivalence relation E is contained in an ergodic, hyperfinite equivalence relation F.*

Granting this, let $S \in \text{ERG}$ be such that $[T] \subseteq [S]$, where for $T \in \text{Aut}(X,\mu)$ we denote by $[T] = [E_T]$ the full group of the equivalence relation E_T induced by T. Now $[S]$ is dense in $(\text{Aut}(X,\mu), w)$. Then repeat the argument in the proof of 5.1, starting with $g \in [S]$.

Proof of the Lemma. Consider the ergodic decomposition of E. This is given by a Borel E-invariant surjection $\pi : X \to \mathcal{E}$, where $\mathcal{E}$ is the space of ergodic invariant measures for E. If $X_e = \{x \in X : \pi(x) = e\}$, then e is the unique invariant ergodic measure for $E|X_e$ and if $\nu = \pi_*\mu$, a probability measure on $\mathcal{E}$, then $\mu = \int e d\nu(e)$, i.e., for $A \subseteq X$ Borel,

$$\mu(A) = \int e(A \cap X_e) d\nu(e).$$

In particular, modulo μ-null sets, every E-invariant set A is of the form $\pi^{-1}(B)$, for some $B \subseteq \mathcal{E}$.

Case 1. *$\mathcal{E}$ is finite.*

We will prove then the result by induction on $\text{card}(\mathcal{E})$. If $\text{card}(\mathcal{E}) = 1$, there is nothing to prove. Assume now $\text{card}(\mathcal{E}) = 2$, say $\mathcal{E} = \{e_1, e_2\}$, with $\mu(X_{e_1}) \geq \mu(X_{e_2})$. Note that $e_i = (\mu|X_{e_i})/\mu(X_{e_i})$. By Dye's Theorem, we can find $X_1 \subseteq X_{e_1}$, with $\mu(X_1) = \mu(X_{e_2})$ and a Borel isomorphism φ from $E|X_1$ to $E|X_{e_2}$ (modulo null sets) which preserves μ. Then define F to be the equivalence relation generated by E and φ. Clearly F is measure preserving and ergodic. Since every F-class consists of at most 2 E-classes, F is hyperfinite.

Assume now that the result is true up to n and say $\mathcal{E} = \{e_1, \dots, e_n, e_{n+1}\}$. By induction hypothesis we can find a hyperfinite measure preserving ergodic equivalence relation E_n on $X_{e_1} \cup \dots \cup X_{e_n}$ extending $E|(X_{e_1} \cup \dots \cup X_{e_n})$. Then if $F_n = E \cup E_n$, F_n is hyperfinite, measure preserving and has 2 ergodic invariant measures, so by the $n = 2$ case, we can find $F \supseteq F_n$, F hyperfinite, ergodic and measure preserving.

Case 2. *$\mathcal{E}$ is countably infinite.*

Say $\mathcal{E} = \{e_1, e_2, \dots)$. By Case 1, there is hyperfinite, measure preserving, ergodic E_n on $X_{e_1} \cup \dots \cup X_{e_n}$ with $E|X_{e_1} \cup \dots \cup E|X_{e_n} \subseteq E_n$ and $E_n \subseteq E_{n+1}$. Let $F = \bigcup_n E_n$. This clearly works.

Case 3. *$\mathcal{E}$ is uncountable and ν is non-atomic.*

Let then $T \in \mathrm{Aut}(\mathcal{E}, \nu)$ be ergodic. For each $e \in \mathcal{E}$, let $\varphi_e : X_e \to X_{T(e)}$ be a Borel bijection such that $\varphi(e, x) = \varphi_e(x)$ is Borel and φ_e is an isomorphism of $(E|X_e, e)$ with $(E|X_{T(e)}, T(e))$ modulo null sets. Let then $\varphi(x) = \varphi_{\pi(x)}(x)$ (so if $x \in X_e, \varphi(x) = \varphi_e(x)$). Note first that $\varphi \in \mathrm{Aut}(X, \mu)$. To see this fix a Borel set $A \subseteq X$. Then

$$\begin{aligned} \mu(\varphi^{-1}(A)) &= \int e(\varphi^{-1}(A) \cap X_e) d\nu(e) \\ &= \int T(e)(\varphi_e(\varphi^{-1}(A) \cap X_e)) d\nu(e) \\ &= \int T(e)(A \cap X_{T(e)}) d\nu(e) \\ &= \int e(A \cap X_e) d\nu(e) \\ &= \mu(A), \end{aligned}$$

as ν is T-invariant.

Let F be the equivalence relation induced by E and φ, so that $E \subseteq F$ and F is measure preserving, ergodic. Finally, E is hyperfinite (μ-a.e.) as every F class can be (Borel uniformly) ordered in order type ζ^2 (where ζ is the order type of $\mathbb{Z}$) (see Jackson-Kechris-Louveau [JKL], Section 2).

Case 4. *$\mathcal{E}$ is uncountable but ν has atoms.*

Let then $\{e_1, e_2, \dots\}$ be the atoms of ν. Let $X_0 = \bigcup_i X_{e_i}, X_1 = X \setminus X_0$. By applying the preceding cases to $(E|X_i, (\mu|X_i)/\mu(X_i)), i = 0, 1$, we see that we can find measure preserving, ergodic, hyperfinite F_i on X_i such that $E|X_i \subseteq F_i$. If $F' = F_1 \cup F_2$, then F is measure preserving, hyperfinite and $E \subseteq F'$. Then, by Case 1 again, we can find measure preserving, ergodic, hyperfinite $F \supseteq F'$ and the proof is complete. □

Similar arguments also show the following.

Theorem 5.5. *Let E be an ergodic equivalence relation. Then the conjugacy action of* $([E], u)$ *on* $(\mathrm{Aut}(X, \mu), w)$ *is generically turbulent.*

Proof. Use the argument in 5.1 to show that every ergodic $T \in [E]$ is turbulent for this action (since the conjugacy class of T in $\mathrm{Aut}(X, \mu)$ is weakly dense, this shows that the action is generically turbulent). □

(B) Given equivalence relations E, F on standard Borel spaces X, Y, we say that E can be *Borel reduced* to F, in symbols

$$E \leq_B F,$$

if there is a Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

We say that E is *Borel bireducible* to F, in symbols

$$E \sim_B F,$$

if $E \leq_B F$ and $F \leq_B E$. An equivalence relation E on a standard Borel space X can be *classified by countable structures* if there is a countable language L and a Borel map $f : X \to X_L$, where X_L is the standard Borel space of countable structures for L (with universe $\mathbb{N}$), such that $xEy \Leftrightarrow f(x) \cong f(y)$, where $\cong$ is the isomorphism relation for structures, i.e, E is Borel reducible to the isomorphism relation of the countable structures of some countable language. Hjorth [Hj2] has shown that if E is induced by a (generically) turbulent action, then E restricted to any dense G_δ set cannot be classified by countable structures.

From 5.3 and the fact that WMIX is dense G_δ one can of course derive that conjugacy in WMIX cannot be classified by countable structures. Earlier such a result for ERG was proved in Hjorth [Hj1]. Moreover, by also using 2.5, it also follows that unitary equivalence in WMIX cannot be classified by countable structures.

Theorem 5.6 (Hjorth [Hj1] for ERG, Foreman-Weiss [FW]). *Conjugacy and unitary (spectral) equivalence in* WMIX *cannot be classified by countable structures.*

In fact one can prove stronger results which apply as well to MIX (which is meager in $\mathrm{Aut}(X, \mu)$).

Fix an uncountable standard Borel space Y and let $P(Y)$ be the standard Borel space of Borel probability measures on Y. As usual call $\mu, \nu \in P(Y)$ *equivalent measures* (or mutually absolutely continuous) if they have the same null sets. Thus denoting equivalence of μ, ν by $\mu \sim \nu$ and absolute continuity by $\mu \ll \nu$, we have

$$\mu \sim \nu \Leftrightarrow \mu \ll \nu \text{ and } \nu \ll \mu.$$

Clearly, up to Borel isomorphism, $\sim$ is independent of the choice of Y. It is known that $\sim$ is a Borel equivalence relation which cannot be classified by countable structures, see Kechris-Sofronidis [KS]. The spectral theorem for unitary operators implies that if $U(H)$ is the unitary group of a separable infinite dimensional space H, then conjugacy in $U(H)$ is Borel bireducible to $\sim$. It is clear then that unitary (spectral) equivalence in $\mathrm{Aut}(X, \mu)$ is Borel reducible to measure equivalence $\sim$.

Theorem 5.7 (Kechris). *(a) Measure equivalence is Borel bireducible to unitary equivalence in* MIX (*and thus also in* WMIX, ERG *and* $\mathrm{Aut}(X, \mu)$)*. In particular, unitary equivalence in* MIX *cannot be classified by countable structures.*

(b) Measure equivalence is Borel reducible to conjugacy in MIX (*and thus also in* WMIX, ERG *and* $\mathrm{Aut}(X, \mu)$)*. In particular, conjugacy in* MIX *cannot be classified by countable structures.*

Our proof will use the spectral theory of unitary operators and measure preserving transformations, for which some standard references are Cornfeld-Fomin-Sinai [CFS], Glasner [Gl2], Goodson [Goo], Lemańczyk [Le], Nadkarni [Na], Parry [Pa], Queffélec [Qu], and in particular the spectral theory of the shift in Gaussian spaces (see Appendices C-E) for which we refer the reader to Cornfeld-Fomin-Sinai [CFS]. Moreover, we will also make use of some results in the harmonic analysis of measures on $\mathbb{T}$, for which we refer to Kechris-Louveau [KL].

Proof of 5.7. (a) For each finite (positive) Borel measure σ on $\mathbb{T}$ consider $\hat{\sigma} : \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{\sigma}(n) = \int \bar{z}^n d\sigma(z).$$

By Herglotz's theorem (see, e.g., Parry [Pa]) $\sigma \leftrightarrow \hat{\sigma}$ is a 1-1 correspondence between finite Borel measures on $\mathbb{T}$ and positive-definite functions on $\mathbb{Z}$. Note that $\hat{\sigma}$ is real iff σ is symmetric, i.e., invariant under conjugation ($\sigma(A) = \sigma(\bar{A})$, for every Borel set A, where $\bar{A} = \{\bar{z} : z \in A\}$). To each symmetric σ then we can associate the real positive-definite function $\varphi(\sigma) = \hat{\sigma}$ and then the Gaussian space $(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)})$, as in Appendix C, and the corresponding shift, which we will denote by T_σ, i.e., $T_\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n-1})_{n\in\mathbb{Z}}$. It is well-known (see, Cornfeld-Fomin-Sinai [CFS], p. 369) that T_σ is mixing iff σ is a *Rajchman measure*, i.e., $\hat{\sigma}(n) \to 0$ as $|n| \to \infty$.

Consider the Wiener chaos decomposition

$$L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}) = L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}, \mathbb{C}) = \bigoplus_{n\geq 0} H_{\mathbb{C}}^{:n:}$$

discussed in Appendix D, and let $U_{T_\sigma} = U_\sigma$ be the unitary operator on $L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)})$ induced by T_σ. Clearly each $H_{\mathbb{C}}^{:n:}$ is invariant under U_σ. Also note that $H_{\mathbb{C}}^{:0:} = \mathbb{C}$ is the subspace of constants and

$$L_0^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}) = \bigoplus_{n\geq 1} H_{\mathbb{C}}^{:n:}.$$

For each finite measure σ on $\mathbb{T}$ consider the Hilbert space $L^2(\mathbb{T}, \sigma)$ and the unitary operator $V_\sigma(f)(z) = zf(z)$. Suppose now that $\Lambda \subseteq \mathbb{T}$ is an *independent set*, i.e., if $z_1, \dots, z_k \in \Lambda$ are distinct, $n_1, \dots, n_k \in \mathbb{Z}$, and $z_1^{n_1} \cdots z_k^{n_k} = 1$, then $n_1 = \cdots = n_k = 0$. If a symmetric σ is supported by $\Lambda \cup \bar{\Lambda}$, then the n-fold convolution $\sigma^{*n} = \sigma * \cdots * \sigma$ is supported by $(\Lambda \cup \bar{\Lambda})^n$ $(= \{z_1 \dots z_n : z_i \in (\Lambda \cup \bar{\Lambda})\})$. Moreover if σ is non-atomic, then actually σ^{*n} is supported by $(\Lambda \cup \bar{\Lambda})^{*n} = \{z_1 \dots z_n : z_i \in (\Lambda \cup \bar{\Lambda}), z_i \neq z_k, \bar{z}_k, \text{ if } i \neq k\}$ and these sets are pairwise disjoint, so $\sigma = \sigma^{*1}, \sigma^{*2}, \dots$ have pairwise disjoint supports. Now the spectral theory of Gaussian shifts shows that the unitary operator U_σ on $H_{\mathbb{C}}^{:n:}, n \geq 1$, is isomorphic to the unitary operator $V_{\sigma^{*n}}$ on $L^2(\mathbb{T}, \sigma^{*n})$; see Cornfeld-Fomin-Sinai [CFS], Chapter 14, §4, Theorem 1. If σ is also a probability measure, let $\sigma_\infty = \sum_{n=1}^{\infty} \frac{1}{2^n} \sigma^{*n}$. Then $L^2(\mathbb{T}, \sigma_\infty) = \bigoplus_{n \geq 1} K_n$, where $K_n = \{f \in L^2(\mathbb{T}, \sigma_\infty) : f \text{ is } 0 \text{ off } \Lambda^{*n}\}$, each K_n is invariant under V_{σ_∞} and the map $f \in K_n \mapsto \frac{1}{\sqrt{2^n}} f \in L^2(\mathbb{T}, \sigma^{*n})$ is an isomorphism of $V_{\sigma_\infty} | K_n$ with $V_{\sigma^{*n}}$. It follows that $U_\sigma | L_0^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)})$ is isomorphic to V_{σ_∞} on $L^2(\mathbb{T}, \sigma_\infty)$. In the language of spectral theory this says that the maximal spectral type of the operator $U_\sigma^0 = U_\sigma | L_0^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)})$ is equal to σ_∞ and it has multiplicity 1. Since, by the spectral theory, two unitary operators are isomorphic iff their maximal spectral types are equivalent and they have the same multiplicity function, we conclude that whenever σ, τ are non-atomic symmetric probability Borel measures on $\mathbb{T}$ each supported by a set of the form $\Lambda \cup \bar{\Lambda}$, Λ independent, and U_σ^0, U_τ^0 are the unitary operators associated to the respective Gaussian shifts (restricted to the orthogonal of the constant functions), then

$$\sigma_\infty \sim \tau_\infty \Leftrightarrow U_\sigma^0 \cong U_\tau^0,$$

where $\cong$ denotes isomorphism of the unitary operators.

But one can now verify that if σ, τ are supported by the same set $\Lambda \cup \bar{\Lambda}$, where Λ is an independent set, then

$$\sigma \sim \tau \Leftrightarrow \sigma_\infty \sim \tau_\infty.$$

Indeed, $\sigma \sim \tau \Rightarrow \sigma_\infty \sim \tau_\infty$ holds simply because $\sigma << \tau \Rightarrow \sigma * \rho << \tau * \rho$, for any ρ. Conversely, assume that $\sigma_\infty \sim \tau_\infty$. Say σ, τ are both supported by $\Lambda \cup \bar{\Lambda}$. Suppose now $\sigma(A) = 0$ but $\tau(A) > 0$, towards a contradiction. Replacing A by $A \cap (\Lambda \cup \bar{\Lambda})$, we can assume that $A \subseteq (\Lambda \cup \bar{\Lambda})$. Since $\sigma_\infty \sim \tau_\infty >> \tau$, we have $\sigma_\infty(A) > 0$, so as $\sigma(A) = 0, \sigma^{*n}(A) > 0$ for some $n > 1$. Thus $A \cap (\Lambda \cup \bar{\Lambda})^{*n} \neq \emptyset$ and therefore $(\Lambda \cup \bar{\Lambda}) \cap (\Lambda \cup \bar{\Lambda})^{*n} \neq \emptyset$, a contradiction. Thus if σ, τ are supported by $\Lambda \cup \bar{\Lambda}$ for the same independent set Λ, then

$$\sigma \sim \tau \Leftrightarrow U_\sigma^0 \cong U_\tau^0.$$

Note also that

$$U_\sigma^0 \cong U_\tau^0 \Leftrightarrow U_\sigma \cong U_\tau,$$

so we finally have:

If σ, τ are symmetric Rajchman (and thus non-atomic) probability measures supported by $\Lambda \cup \bar{\Lambda}$ for the same independent set Λ, then T_σ, T_τ are mixing and

$$\sigma \sim \tau \Leftrightarrow T_\sigma, T_\tau \text{ are unitarily equivalent.}$$

As $\sigma \mapsto T_\sigma$ is clearly Borel, we have shown that $\sim$ restricted to symmetric Rajchman probability measures supported by $\Lambda \cup \bar{\Lambda}$, for a fixed independent set Λ, is Borel reducible to unitary equivalence on MIX. It only remains to show that $\sim$ can be Borel reduced to $\sim$ restricted to such measures.

We will use here the following result of Rudin [Ru1] (see also Kahane-Salem [KS], VIII, **3**, Théorème II): There is a closed set $\Lambda \subseteq \mathbb{T}$ which is independent and supports a Rajchman measure, i.e., Λ is in the class M_0 of closed sets of *restricted multiplicity* (see Kechris-Louveau [KL]). By a theorem of Kaufman (see Kechris-Louveau [KL], VII. 1, Theorem 7), there is a Borel function $x \mapsto \sigma_x$ from $2^{\mathbb{N}}$ to the standard Borel space of probability measures on $\mathbb{T}$ such that each σ_x is a Rajchman measure, $\text{supp}(\sigma_x) \cap \text{supp}(\sigma_y) = \emptyset$ if $x \neq y$ (where $\text{supp}(\sigma) = \mathbb{T} \setminus \bigcup\{U \text{ open} : \sigma(U) = 0\}$), and $\text{supp}(\sigma_x) \subseteq \Lambda$. Then, since Λ is independent, we have $A \cap \overline{B} = \emptyset$, for disjoint $A, B \subseteq \Lambda$, so, by replacing σ_x by $\frac{1}{2}(\sigma_x + \bar{\sigma}_x)$ (where $\bar{\sigma}(A) = \sigma(\bar{A})$), we have the following conclusion: There is a Borel map $x \mapsto \sigma_x$ from $2^{\mathbb{N}}$ to symmetric Rajchman probability Borel measures with $\text{supp}(\sigma_x) \subseteq \Lambda \cup \bar{\Lambda}$ and $\text{supp}(\sigma_x) \cap \text{supp}(\sigma_y) = \emptyset$ if $x \neq y$.

Now consider an arbitrary probability Borel measure μ on $2^{\mathbb{N}}$. Put

$$\sigma_\mu = \int \sigma_x d\mu(x)$$

i.e., $\sigma_\mu(A) = \int \sigma_x(A) d\mu(x)$, for each Borel set $A \subseteq \mathbb{T}$, or equivalently $\int f d\sigma_\mu = \int (f d\sigma_x) d\mu(x)$ for every continuous function f on $\mathbb{T}$. It follows that

$$\hat{\sigma}_\mu(n) = \int \hat{\sigma}_x(n) d\mu(x)$$

and since $\hat{\sigma}_x(n) = f_n(x) \to 0$ as $|n| \to \infty$ and $|f_n(x)| \leq 1$ we have, by Lebesgue Dominated Convergence, that $\hat{\sigma}_\mu(n) \to 0$, i.e., σ_μ is a Rajchman measure. Clearly σ_μ is symmetric and σ_μ is supported by $\Lambda \cup \bar{\Lambda}$. It only remains to verify that

$$\mu \ll \nu \Leftrightarrow \sigma_\mu \ll \sigma_\nu.$$

Assume $\mu \ll \nu$ and let $A \subseteq \mathbb{T}$ be Borel with $\sigma_\nu(A) = 0$, so that

$$\int \sigma_x(A) d\nu(x) = 0.$$

Then $\sigma_x(A) = 0, \nu$–a.e.(x), thus $\sigma_x(A) = 0, \mu$–a.e.(x), therefore $\sigma_\mu(A) = 0$. Thus $\sigma_\mu \ll \sigma_\nu$. Conversely, assume that $\sigma_\mu \ll \sigma_\nu$ and let $X \subseteq 2^{\mathbb{N}}$ be Borel with $\nu(X) = 0$. Let $X^* = \bigcup_{x \in X} \text{supp}(\sigma_x)$, which is a Borel set in $\mathbb{T}$. Then $\sigma_\nu(X^*) = \int \sigma_x(X^*) d\nu(x) = \nu(X) = 0$. Thus $\sigma_\mu(X^*) = \int \sigma_x(X^*) d\mu(x) = 0$, so $\sigma_x(X^*) = 0, \mu$–a.e.(x), therefore $x \notin X, \mu$–a.e.(x), i.e., $\mu(X) = 0$. So $\mu \ll \nu$.

This concludes the proof of (a).

(b) By the proof in part (a), it is clearly enough to show that if σ, τ are symmetric measures on $\mathbb{T}$, and $\sigma \sim \tau$, then T_σ, T_τ are isomorphic transformations, i.e., there is an isomorphism $S : (\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}) \to (\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\tau)})$ sending T_σ to T_τ.

Suppose $\sigma \sim \tau$. Then the map

$$f \in L^2(\mathbb{T}, \sigma) \mapsto \frac{d\sigma}{d\tau} f \in L^2(\mathbb{T}, \tau)$$

is an isomorphism of the Hilbert space $L^2(\mathbb{T}, \sigma)$ to $L^2(\mathbb{T}, \tau)$ that sends V_σ to V_τ. Since σ, τ are both symmetric, i.e., invariant under conjugation, so is $\frac{d\sigma}{d\tau}$, i.e., $\frac{d\sigma}{d\tau}(z) = \frac{d\sigma}{d\tau}(\bar{z})$ ($\tau-$a.e.). Let $S^2(\mathbb{T}, \sigma)$ be the closed subset of $L^2(\mathbb{T}, \sigma)$ consisting of all $f \in L^2(\mathbb{T}, \sigma)$ satisfying $f(\bar{z}) = \overline{f(z)}$. This is a real subspace of $L^2(\mathbb{T}, \sigma)$ invariant under the operator V_σ and $L^2(\mathbb{T}, \sigma)$ is the complexification of $S^2(\mathbb{T}, \sigma)$, since any $f \in L^2(\mathbb{T}, \sigma)$ is uniquely written as $f_1 + if_2$, with $f_1, f_2 \in S^2(\mathbb{T}, \sigma)$, namely $f_1(z) = \frac{1}{2}[f(z) + \overline{f(\bar{z})}], f_2(z) = \frac{1}{2i}[f(z) - \overline{f(\bar{z})}]$. Moreover, V_σ on $L^2(\mathbb{T}, \sigma)$ is the complexification of $V_\sigma | S^2(\mathbb{T}, \sigma)$. Now the spectral theory of the Gaussian shift T_σ asserts that there is an isomorphism between the real Hilbert space $H_\sigma = H^{:1:}$ (the first chaos of $L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}, \mathbb{R})$ associated with the Gaussian space $(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)})$) and the real Hilbert space $S^2(\mathbb{T}, \sigma)$, which sends $U_\sigma | H_\sigma$ to $V_\sigma | S^2(\mathbb{T}, \sigma)$ (see Cornfeld-Fomin-Sinai [CFS], p. 368). Similarly for τ. Since $\frac{d\sigma}{d\tau}$ is invariant under conjugation, the map $f \in L^2(\mathbb{T}, \sigma) \mapsto \frac{d\sigma}{d\tau} f \in L^2(\mathbb{T}, \tau)$ sends $S^2(\mathbb{T}, \sigma)$ to $S^2(\mathbb{T}, \tau)$ and $V_\sigma | S^2(\mathbb{T}, \sigma)$ to $V_\tau | S^2(\mathbb{T}, \sigma)$. Thus there is an isomorphism T between H_σ and H_τ that sends $U_\sigma | H_\sigma$ to $U_\tau | H_\tau$. By Appendix D, there is an isomorphism $S : (\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}) \to (\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\tau)})$ such that if $O_S(f) = f \circ S^{-1}$ is the corresponding isomorphism of $L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\sigma)}, \mathbb{R})$ with $L^2(\mathbb{R}^{\mathbb{Z}}, \mu_{\varphi(\tau)}, \mathbb{R})$, then $O_S | H_\sigma = T$. Since T sends $U_\sigma | H_\sigma$ to $U_\tau | H_\tau$, S sends T_σ to T_τ and the proof is complete. □

(C) Very recently Foreman, Rudolph and Weiss [FRW] have shown that the conjugacy equivalence relation on ERG is not Borel (in fact it is a complete $\mathbf{\Sigma_1^1}$ subset of ERG^2). This in particular shows that conjugacy on ERG cannot be Borel reduced to unitary equivalence on ERG and thus, in combination with 5.7, it shows that, in terms of Borel reducibility, conjugacy on ERG is strictly more complicated that unitary equivalence on ERG.

Although the place of unitary equivalence on ERG in the Borel reducibility hierarchy of complexity of equivalence relations is completely understood, it is still open to determine this for conjugacy on ERG. For example, how does it compare with the universal equivalence relation induced by a Borel action of $(\mathrm{Aut}(X, \mu), w)$ (or equivalently of $U(H)$, which is isomorphic to a closed subgroup of $(\mathrm{Aut}(X, \mu), w)$, by Appendix E)?

Another question (motivated by a discussion with Yehuda Shalom) on which not much seems to be known is the following. It has been a classical problem of ergodic theory to distinguish up to conjugacy ergodic transformations which are unitarily equivalent. The concept of entropy provided a powerful tool for attacking this problem. In this vein one can raise the general problem of understanding the complexity of conjugacy within each unitary equivalence class. More precisely, let $C \subseteq \mathrm{ERG}$ denote a given unitary equivalence class in ERG. How complicated is conjugacy restricted to C? The answer will depend of course on C. If C corresponds to a discrete spectrum ergodic transformation T (i.e., one for which U_T has discrete spectrum, which means that the corresponding maximal spectral type concentrates on a countable set), then, by the classical Halmos-von Neumann Theorem, C consists of a single conjugacy class. On the other hand not much seems to be known about the complexity of conjugacy for C whose maximal spectral type is continuous (i.e., non-atomic). A particular case of interest is the class C which has countable homogeneous Lebesgue spectrum (i.e., the unitary equivalence class of the shift on $2^{\mathbb{Z}}$ with the usual product measure), which of course contains continuum many conjugacy classes.

Remark. Concerning the spectral theory of ergodic measure preserving transformations, it is a very hard and still unsolved problem to determine what are the possible spectral invariants, i.e., pairs of measure classes in $P(\mathbb{T})$ and multiplicity functions, that correspond to (the unitary operator associated to) an ergodic measure preserving transformation. Equivalently, this can be viewed as the question of characterizing the unitary operators that are realized by ergodic, measure preserving transformations up to isomorphism (i.e., conjugacy in the unitary group) - see here also Appendix H, (**F**). Fox example, one can ask if the set of such of operators is Borel in $U(H)$. (It is obviously $\mathbf{\Sigma}^1_1$.) It also appears to be unknown what measure classes of probability measures on $\mathbb{T}$ can appear as maximal spectral types of ergodic measure preserving transformations. (See, e.g., Katok-Thouvenot [KTh].) For example, one can ask whether the set of maximal spectral types of such transformations is a Borel set in $P(\mathbb{T})$. (It is clearly $\mathbf{\Sigma}^1_1$.)

6. Automorphism groups of equivalence relations

(A) For each measure preserving countable Borel equivalence relation E on (X, μ) we denote by $N[E]$ the group of all $T \in \mathrm{Aut}(X, \mu)$ such that

$$xEy \Leftrightarrow T(x)ET(y),$$

for all x, y in a conull set. Note that $N[E]$ is the normalizer of $[E]$ in $\mathrm{Aut}(X, \mu)$. If E is not smooth, then T.-J. Wei [We] showed that $N[E]$ is $\mathbf{\Pi}^0_3$-complete in $(\mathrm{Aut}(X, \mu), w)$. Clearly $N[E]$ is closed in $(\mathrm{Aut}(X, \mu), u)$ but it may not be separable. For example, if $E = E_0$ and for $A \subseteq \mathbb{N}$ we let $f_A(x) = x + \chi_A$, where addition is pointwise modulo 2, then $f_A \in N[E]$ and $\delta_u(f_A, f_B) = 1$ if $A \neq B$. Next we will see that $N[E]$ is a Polishable subgroup of $(\mathrm{Aut}(X, \mu), w)$, if E is aperiodic.

Note that each $T \in N[E]$ induces by conjugation an isometry i_T of $([E], \delta_u) : i_T(S) = TST^{-1}$. Consider the Polish group $\mathrm{Iso}([E], \delta_u)$, with the pointwise convergence topology and the map $i(T) = i_T$. We first check that it is an algebraic isomorphism of $N[E]$ with a subgroup of $\mathrm{Iso}([E], \delta_u)$, provided that E is aperiodic. It is clearly a homomorphism. Injectivity follows from 4.11.

We next verify that the image $i(N[E])$ of $N[E]$ is closed in $\mathrm{Iso}([E], \delta_u)$, provided E is aperiodic.

To see this let $T_n \in N[E]$ and assume $i_{T_n} \to i_0$ in $\mathrm{Iso}([E], \delta_u)$. Then for every $S \in [E], \{T_n S T_n^{-1}\}$ is δ_u-Cauchy, i.e.,

$$\lim_{m,n\to\infty} \delta_u((T_m^{-1}T_n)S(T_m^{-1}T_n)^{-1}, S) \to 0.$$

Fix $A \in \mathrm{MALG}_\mu$ and find $S \in [E]$ with $\mathrm{supp}(S) = A$ (by 4.10). Since $\mathrm{supp}((T_m^{-1}T_n)S(T_m^{-1}T_n)^{-1}) = T_m^{-1}T_n(A)$, we have $\mu(T_m^{-1}T_n(A)\Delta A) \to 0$, as $m, n \to \infty$. Similarly, since also $(i_{T_n})^{-1} = i_{T_n^{-1}} \to i_0^{-1}$ in $\mathrm{Iso}([E], \delta_u)$, we have $\lim_{m,n\to 0} \mu(T_m T_n^{-1}(A)\Delta A) = 0$. So $\lim_{m,n\to\infty}[\mu(T_n(A)\Delta T_m(A)) + \mu(T_n^{-1}(A)\Delta T_m^{-1}(A)] \to 0$, i.e., $\{T_n\}$ is a Cauchy sequence in the complete metric $\bar{\delta}_w$ of $\mathrm{Aut}(X, \mu)$. Thus there is $T \in \mathrm{Aut}(X, \mu)$ with $T_n \to T$ weakly. Then for $S \in [E], T_n S T_n^{-1} \to TST^{-1}$ weakly. But also $T_n S T_n^{-1} \to i_0(S)$ uniformly, so $TST^{-1} = i_0(S) \in [E]$. Thus $T \in N[E]$ and clearly $i_T = i_0$.

We summarize the preceding discussion as follows.

Theorem 6.1. *Let E be aperiodic. The group $N[E]$ is a Polishable subgroup of $(\mathrm{Aut}(X, \mu), w)$. The corresponding Polish topology $\tau_{N[E]}$ is given by the complete metric*

$$\delta_{N[E]}(T_1, T_2) = \sum_n 2^{-n}[\delta_u(T_1 S_n T_1^{-1}, T_2 S_n T_2^{-1}) + \delta_u(T_1^{-1} S_n T_1, T_2^{-1} S_n T_2)],$$

where $\{S_n\}$ is dense in $([E], u)$. We have $w|N[E] \subseteq \tau_{N[E]} \subseteq u|N[E]$, and thus if $T_n \to T$ in $N[E]$, then $T_n \to T$ weakly.

Convention. *From now on, when we consider topological properties of $N[E]$ without explicitly indicating the topology, we will always assume that we refer to $\tau_{N[E]}$.*

If $E = E_\Gamma^X$, where the countable group Γ acts in a Borel way on X, we also have the following fact, denoting for each $\gamma \in \Gamma$ also by γ the map $\gamma(x) = \gamma \cdot x$.

Proposition 6.2. *For $E = E_\Gamma^X$ aperiodic, $T_n, T \in N[E]$,*

$$T_n \to T \Leftrightarrow T_n \to T \text{ weakly and } \forall \gamma \in \Gamma(T_n \gamma T_n^{-1} \to T\gamma T^{-1} \text{ uniformly}).$$

Proof. Again it is enough to show that if $T_n \in N[E], T_n \to 1$ weakly and $T_n \gamma T_n^{-1} \to \gamma$ uniformly, $\forall \gamma \in \Gamma$, then $T_n T T_n^{-1} \to T$ uniformly, $\forall T \in [E]$.

Fix $T \in [E]$. Then there is a partition $X = \bigcup_i A_i, A_i$ Borel, and $\gamma_i \in \Gamma$ with $T(x) = \gamma_i(x), \forall x \in A_i$. Fix $\epsilon > 0$. Choose then M large enough so that

$\sum_{i>M} \mu(A_i) < \epsilon$. Then for some N and all $n \geq N$,

$$\mu(T_n(A_i)\Delta A_i) < \frac{\epsilon}{(M+1)}, \forall i \leq M,$$

$$T_n \gamma_i T_n^{-1}(x) = \gamma_i(x), \forall i \leq M,$$

for all $x \notin A_n^i$, where $\mu(A_n^i) < \frac{\epsilon}{(M+1)}, i \leq M$. Thus off a set of measure $< 2\epsilon$, we have that if $i \leq M$ and $x \in A_i$, then $T_n^{-1}(x) \in A_i$ and

$$TT_n^{-1}(x) = \gamma_i(T_n^{-1}(x)) = T_n^{-1}(\gamma_i(x)) = T_n^{-1}T(x).$$

Thus $\mu(\{x : TT_n^{-1}(x) \neq T_n^{-1}T(x)\}) < 3\epsilon$, if $n \geq N$, so $\delta_u(TT_n^{-1}, T_n^{-1}T) = \delta_u(T_nTT_n^{-1}, T) \to 0$. □

Thus the following is also a complete metric for $N[E]$:

$$\bar{\delta}_{N[E]}(T_1, T_2) = \bar{\delta}_w(T_1, T_2) + \sum_n 2^{-n}\delta_u(T_1\gamma_n T_1^{-1}, T_2\gamma_n T_2^{-1}),$$

where $\Gamma = \{\gamma_n\}$.

Remark. It is clear that in the preceding one can equivalently use δ'_u instead of δ_u.

(B) There is also another way to understand the topology of $N[E]$. Consider the measure M on E defined by

$$M(A) = \int \text{card}(A_x) d\mu(x),$$

where $A_x = \{y : (x, y) \in A\}$, for any Borel set $A \subseteq E$. This is a σ-finite Borel measure on E.

Given now $T \in N[E]$ consider the map $T \times T$ on E defined by

$$T \times T(x, y) = (T(x), T(y)).$$

Then it is easy to see that $T \times T$ preserves the measure M, therefore induces a unitary operator on the Hilbert space

$$L^2(E, M)$$

denoted by $U_{T\times T}$:

$$U_{T\times T}(f)(x, y) = f(T^{-1}(x), T^{-1}(y)).$$

The map $T \mapsto U_{T\times T}$ is a group isomorphism between $N[E]$ and a subgroup of $U(L^2(E, M))$. (To see that it is 1-1, assume that $U_{T\times T} = 1$, i.e., $f(T^{-1}(x), T^{-1}(y)) = f(x, y)$, M-a.e., for all $f \in L^2(E, M)$. Then for any $g \in L^2(X, \mu)$, if

$$f(x, y) = \begin{cases} g(x), & \text{if } x = y, \\ 0, & \text{otherwise}, \end{cases}$$

then $f \in L^2(E, M)$ and $g(T^{-1}(x)) = f(T^{-1}(x), T^{-1}(x)) = f(x, x) = g(x)$, μ-a.e., so $T = 1$.)

Now notice that $U_{T_n\times T_n} \to 1$ in $U(L^2(E, M)) \Rightarrow T_n \to 1$ in $N[E]$. This is because for any $T \in N[E], S \in [E]$, if f is the characteristic function of graph(S) $\subseteq E$, then $\delta_u(TST^{-1}, S) \leq \|U_{T\times T}(f) - f\|_2^2$. On the other hand $T \mapsto U_{T\times T}$ is clearly a Borel homomorphism from the Polish group $N[E]$ into the Polish group $U(L^2(E, M))$, thus it is continuous, so if $T_n \to 1$ in $N[E]$, then $U_{T_n\times T_n} \to 1$ in $U(L^2(X, M))$. So $T \mapsto U_{T\times T}$ is a homeomorphism of $N[E]$ with a (necessarily closed) subgroup of $U(L^2(X, M))$, and we have the following fact.

Proposition 6.3. *For aperiodic E, the map $T \mapsto U_{T\times T}$ from $N[E]$ to $U(L^2(E, M))$ is a (topological group) isomorphism of $N[E]$ with a closed subgroup of $U(L^2(E, M))$.*

Thus we can identify each $T \in N[E]$ with the corresponding unitary operator $U_{T\times T}$ on $L^2(E, M)$.

(C) It follows from 4.1 that for every ergodic $E, N[E]$ can be also identified with the automorphism group, Aut($[E]$), of the (abstract) group $[E]$. More precisely, every $S \in N[E]$ gives rise to the automorphism $T \mapsto STS^{-1}$ of $[E]$ and every automorphism of the (abstract) group $[E]$ is of this form for a unique $S \in N[E]$. Next notice that $[E]$ is the smallest non-trivial normal subgroup of $N[E]$. Indeed, if $K \lhd N[E]$ is another normal subgroup, then, since $[E]$ is simple (by 4.6), either $[E] \leq K$ or else $[E] \cap K = \{1\}$. In the latter case $K \subseteq C_{[E]}$, the centralizer of $[E]$ in Aut(X, μ), which by 4.11 is trivial, so $K = \{1\}$.

Assume now E, F are ergodic and $f : N[E] \to N[F]$ is an (abstract) group isomorphism. Then f sends $[E]$ onto $[F]$ so, by 4.1, $E \cong F$ and there is $\varphi \in \text{Aut}(X, \mu)$ such that $f(T) = f_\varphi(T) = \varphi T\varphi^{-1}, \forall T \in [E]$. Thus f_φ maps $[E]$ onto $[F]$ and therefore f_φ maps $N[E]$ onto $N[F]$. Consider $f^{-1} \circ f_\varphi$. This is an automorphism of $N[E]$ which is trivial on $[E]$. Since $[E]$ is simple, non-abelian, a theorem of Burnside (see Thomas [Tho], 1.2.8) asserts that every automorphism of $N[E]$ (which can be identified with the automorphism group of $[E]$) is inner and thus there is $\psi \in N[E]$ such that $f^{-1} \circ f_\varphi(T) = \psi T\psi^{-1}, \forall T \in N[E]$, and as $f^{-1} \circ f_\varphi(T) = T$ for $T \in [E], \psi T\psi^{-1} = T, \forall T \in [E]$, i.e., $\psi \in C_{[E]}$, so $\psi = 1$. Thus $f(T) = f_\varphi(T), \forall T \in N[E]$.

So we have the following Reconstruction Theorem for $N[E]$.

Theorem 6.4. *If E, F are ergodic equivalence relations, the following are equivalent:*

(i) $E \cong F$,

(ii) $N[E], N[F]$ are isomorphic as abstract groups.

Moreover, for any algebraic isomorphism $f : N[E] \to N[F]$ there is unique $\varphi \in \text{Aut}(X, \mu)$ with $f(T) = \varphi T\varphi^{-1}, \forall T \in N[E]$.

(D) Finally, we note that Danilenko [Da] has shown that (as a topological group) $N[E]$ is contractible, when E is ergodic, hyperfinite. We will see some further properties of such $N[E]$ in the next section.

Comments. For the definition of the topology on $N[E]$, see Hamachi-Osikawa [HO], Gefter [Ge], Danilenko [Da].

7. The outer automorphism group

(A) The group $[E]$ is normal in $N[E]$ and the quotient

$$\mathrm{Out}(E) = N[E]/[E]$$

is called the *outer automorphism group* of E. If E is aperiodic and $N[E]$ has the Polish topology $\tau_{N[E]}$, then $\mathrm{Out}(E)$ with the quotient topology will be a Polish group iff $[E]$ is closed in $N[E]$.

The group $[E]$ may or may not be closed in $N[E]$ (we will say more about this below) but it is always a Polishable subgroup of $N[E]$ with the corresponding Polish topology being equal to the uniform topology on $[E]$. This is because $\tau_{N[E]}|[E] \subseteq u|[E]$. Moreover we have that $[E]$ will be closed in $N[E]$ exactly when $\tau_{N[E]}|[E] = u|[E]$, i.e., for $T_n \in [E]$, $T_n S T_n^{-1} \to S$ uniformly, $\forall S \in [E]$, implies $T_n \to 1$ uniformly. If $E = E_\Gamma^X$, this is also equivalent to the assertion that $T_n \to 1$ weakly and $\forall \gamma \in \Gamma (T_n \gamma T_n^{-1} \to \gamma$ uniformly) implies $T_n \to 1$ uniformly.

Proposition 7.1. *If E is ergodic and $\{g_n\}$ is dense in* $([E], u)$*, then*

$$[E] = N[E] \cap \bigcup_n \{T : \delta_u(T, g_n) < 1\}.$$

In particular, $[E]$ *is open in* $(N[E]), u)$ *and an F_σ Polishable subgroup of* $(N[E], \tau_{N[E]})$.

Proof. If $T \in N[E]$ and $\delta_u(T, g_n) < 1$, then $T(x) = g_n(x)$ and thus $T(x)Ex$ on a positive measure set. Then, by the ergodicity of $E, T(x)Ex$, a.e., i.e., $T \in [E]$.

Since every open ball in u is F_σ in w and $w|N[E] \subseteq \tau_{N[E]}$, it follows that $[E]$ is F_σ in $(N[E], \tau_{N[E]})$. □

The preceding result should be contrasted with the fact that, for ergodic E, $[E]$ is $\mathbf{\Pi}_3^0$-complete in $(\mathrm{Aut}(X, \mu), w)$ (see the paragraph after 3.1).

(B) We next study the complexity of $\mathrm{Out}(E)$, when E is ergodic, hyperfinite.

Theorem 7.2 (Hamachi-Osikawa [HO]). *If E is ergodic, hyperfinite, then $[E]$ is dense in $N[E]$.*

Proof. Take $E = E_0$ on $2^{\mathbb{N}}$. Consider the basic open sets $\{N_s\}_{s \in 2^n}$ and let G_n be the permutation group of 2^n. We view each $g \in G_n$ as a member of $[E_0]$ (a dyadic permutation) by letting $g(s\hat{\ }x) = g(s)\hat{\ }x$. So $G_1 \subseteq G_2 \subseteq \dots$ and $G_\infty = \bigcup_n G_n$ is dense in $([E_0], u)$. Finally note that if for $s \in 2^n, g_s$ is the transposition $g_s = (0^n\ s)$ that switches $0^n, s$ (with $g_{0^n} = \mathrm{id}$), then G_n is generated by $\{g_s\}_{s \in 2^n}$ (as $(a\ b) = (x\ a)(x\ b)(x\ a)$).

If $T \in N[E_0], n \geq 1$, we will find $T_n \in [E_0]$ such that

$$T_n(N_s) = T(N_s), T_n g_s T_n^{-1} = T g_s T^{-1},$$

for any $s \in 2^n$. Then $T_n g T_n^{-1} = T g T^{-1}$ for any $g \in G_n$, so $T_n g T_n^{-1} \to T g T^{-1}$ uniformly for any $g \in G_\infty$, and $T_n \to T$ weakly, thus $T_n \to T$ in $N[E]$.

Definition of T_n: Let $R \in [E_0]$ be such that $R(N_{0^n}) = T(N_{0^n})$. Then define T_n on $N_s, s \in 2^n$, as follows:

$$(*) \qquad T_n(x) = T g_s T^{-1} R g_s(x).$$

As $g_s, T g_s T^{-1}, R \in [E_0]$, clearly $T_n \in [E_0]$. Also $T_n | N_{0^n} = R | N_{0^n}, T_n(N_s) = T(N_s)$. We now check that $T_n g_s T_n^{-1}(y) = T g_s T^{-1}(y), \forall s \in 2^n, \forall y$. If $y \notin T(N_s) \cup T(N_{0^n}), T g_s T^{-1}(y) = T T^{-1}(y) = y$ and $T_n g_s T_n^{-1}(y) = T_n T_n^{-1}(y) = y$. If $y \in T(N_s) \cup T(N_{0^n})$ this follows from (*). □

Since clearly $[E_0] \neq N[E_0]$ (e.g., any infinite $A \subseteq \mathbb{N}$ induces an element of $N[E_0] \setminus [E_0]$ via $x \mapsto x + \chi_A (\text{mod } 2)$), it follows that for any ergodic, hyperfinite E, $\mathrm{Out}(E)$ is not Polish. In fact one can say quite a bit more.

Call $\mathrm{Out}(E)$ *turbulent* if the action of $([E], u)$ by (right) translation on $N[E]$ is turbulent. In that case, the equivalence relation induced by the cosets of $[E]$ in $N[E]$ admits no classification by countable structures (i.e., $\mathrm{Out}(E)$ cannot be Borel embedded in the isomorphism types of countable structures).

Theorem 7.3 (Kechris). *If E is not E_0-ergodic, then every element of $\overline{[E]}$ (closure of $[E]$ in $N[E]$) is turbulent for the translation action of $([E], u)$ on $N[E]$. In particular, if also $[E]$ is not closed in $N[E]$, the translation action of $([E], u)$ on $\overline{[E]}$ is turbulent and the associated coset equivalence relation admits no classification by countable structures.*

Corollary 7.4. *If E is ergodic, hyperfinite, then* $\mathrm{Out}(E)$ *is turbulent.*

Proof. Let $G = \overline{[E]}$. Since the periodic $g \in [E]$ are dense in $([E], u)$, fix a countable set $\{g_j\}$ of periodic elements in $[E]$ which is dense in $([E], u)$. Then a basic nbhd of $g \in G$ has the form

$$U = \bigcap_{j=1}^{n} \{S \in G : \delta'_u(S g_j S^{-1}, g g_j g^{-1}) < \epsilon\}.$$

Also fix a basic nbhd V of 1 in $([E], u)$.

We will show that $\mathcal{O}(g, U, V)$ is dense in U. Now $\{gT^{-1} : T \in [E]\}$ is dense in G, so $\{gT^{-1} \in U : T \in [E]\}$ is dense in U. Therefore $\{g g_j^{-1} \in U : j = 0, 1, \dots\}$ is dense in U, thus it is enough, for each $\rho > 0$, and $h \in \{g_j\}$ such that $gh^{-1} \in U$, to show that $\mathcal{O}(g, U, V) \cap \{S : \delta'_u(S, gh^{-1}) < \rho\} \neq \emptyset$. Note that for $S \in G$:

$$\begin{aligned} gS \in U &\Leftrightarrow \forall j \leq n [\delta'_u(g S g_j S^{-1} g^{-1}, g g_j g^{-1}) < \epsilon] \\ &\Leftrightarrow \forall j \leq n [\delta'_u(S g_j S^{-1}, g_j) < \epsilon]. \end{aligned}$$

It is thus enough, for each $\rho > 0$, to find a continuous path $\lambda \mapsto h_\lambda$ in $([E], u)$ with $h_0 = 1$ such that $\delta'_u(h_\lambda^{-1} g_j h_\lambda, g_j) < \epsilon, \forall j \leq n$, and $\delta'_u(h, h_1) < \rho$. Let $\alpha < \epsilon$ be such that $\delta'_u(h^{-1} g_j h, g_j) < \alpha, \forall j \leq n$. Using the notation of 5.2, we can find $\bar{h}, \bar{g}_j, j \leq n$, which are in $[\pi^{-1}(E_N) \cap E]$ for some large enough N and $\delta'_u(h, \bar{h}) < \delta_0, \delta'_u(g_j, \bar{g}_j) < \delta_0, \forall j \leq n$, where $0 < \delta_0 < \rho, \alpha + 6\delta_0 < \epsilon$. Then $\delta'_u((\bar{h})^{-1} \bar{g}_j \bar{h}, \bar{g}_j) < 3\delta_0 + \alpha + \delta_0 = \alpha + 4\delta_0$. Then, as in the proof of

5.2, there is a continuous path $\lambda \mapsto h_\lambda$ in $([E], u)$ such that $h_0 = 1, h_1 = \bar{h}$ and $\delta'_u(h_\lambda^{-1}\bar{g}_j h_\lambda, \bar{g}_j) < \alpha + 4\delta_0, \forall j \leq n$. Then $\delta'_u(h, h_1) = \delta'_u(h, \bar{h}) < \delta_0 < \rho, \delta'_u(h_\lambda^{-1} g_j h_\lambda, g_j) < 2\delta'_u(g_j, \bar{g}_j) + \alpha + 4\delta_0 \leq \alpha + 6\delta_0 < \epsilon$. □

Remark (Tsankov). As we noted earlier, $N[E]$ is closed in the space $(\mathrm{Aut}(X, \mu), u)$ and so δ_u is a complete metric on $N[E]$. Also as we saw in the proof of 7.1 $\delta_u(S, T) = 1$ if $S \in [E], T \in N[E] \setminus [E]$, so for any two distinct cosets $[E]S \neq [E]T$ of $[E]$ in $N[E], \delta_u(S', T') = 1, \forall S' \in [E]S, \forall T' \in [E]T$. It follows that $(N[E], u)$ is separable (i.e., Polish) iff $\mathrm{Out}(E)$ is countable iff $\tau_{N[E]} = u|N[E]$.

Remark. Consider the *shift action* of a countable group Γ on (X^Γ, μ^Γ), given by $\gamma \cdot p(\delta) = p(\gamma^{-1}\delta)$, with corresponding equivalence relation E. Any $S \in \mathrm{Aut}(X, \mu)$ induces $S^* \in \mathrm{Aut}(X^\Gamma, \mu^\Gamma)$ given by

$$S^*(p) = (\gamma \mapsto S(p(\gamma)).$$

Clearly $S^* \in N[E]$, in fact $S^*\gamma = \gamma S^*, \forall \gamma \in \Gamma$. It is easy to check that $\delta_u(S^*, T) = 1$, for every $S \neq 1, T \in [E]$. It follows that $S \mapsto S^*$ is a Borel, therefore, continuous embedding of $(\mathrm{Aut}(X, \mu), w)$ into $N[E]$ with image having trivial intersection with $[E]$. In fact, it is a homeomorphism of $(\mathrm{Aut}(X, \mu), w)$ with (a necessarily closed) subgroup of $N[E]$ since, in view of 6.2, $S_n^* \to S^*$ in $N[E]$ iff $S_n^* \to S^*$ weakly and this is easily equivalent to $S_n \to S$ weakly. If $\pi : N[E] \to \mathrm{Out}(E)$ is the projection map and $G = \{S^* : S \in \mathrm{Aut}(X, \mu)\}$, then it is easy to check that $\pi|G$ is a homeomorphism, so $(\mathrm{Aut}(X, \mu), w)$ embeds topologically as a subgroup of $\mathrm{Out}(E)$. So, if moreover $\mathrm{Out}(E)$ is Polish, this copy of $(\mathrm{Aut}(X, \mu), w)$ in $\mathrm{Out}(E)$ is closed, i.e., $[E]G$ is closed in $N[E]$. Finally note that for any distinct $T_1, T_2 \in \mathrm{Aut}(X, \mu), \delta_u(T_1^*, T_2^*) = 1$, i.e., G is discrete in u. In particular, taking $\Gamma = \mathbb{Z}$, we see that if E is ergodic, hyperfinite, then $N[E]$ contains a closed subgroup G isomorphic to $(\mathrm{Aut}(X, \mu), w)$ with $G \cap [E] = \{1\}$ and moreover G is discrete in u.

Remark. Assume now that E is ergodic, hyperfinite. Then Connes and Krieger [CK] have shown that $\mathrm{Out}(E)$ has the following very strong property: Every two elements of $\mathrm{Out}(E)$ which have the same order (finite or infinite) are conjugate. Since, by the preceding remark, $\mathrm{Out}(E)$ contains a copy of $\mathrm{Aut}(X, \mu)$, which is simple and has elements of any order, it follows that $\mathrm{Out}(E)$ is simple. Also since in $\mathrm{Aut}(X, \mu)$ every element is a commutator and a product of three involutions (see Section 2, **(D)**) the same is true for $\mathrm{Out}(E)$. Going up to $N[E]$ we see that $[E]$ is the unique non-trivial normal subgroup of $N[E]$. Moreover, using again the preceding remark and the result of Connes-Krieger, we see that there is a copy G of $\mathrm{Aut}(X, \mu)$ in $N[E]$ so that for every $T \in N[E]$ some conjugate of T is of the form $S_1 S_2$, where $S_1 \in [E]$ and $S_2 \in G$. It follows (using also 2.10 and the comments in Section 4) that every element of $N[E]$ is the product of 2 commutators and 6 involutions. I do not know if every (algebraic) automorphism of $\mathrm{Out}(E)$

is inner. Also I do not know whether the following is true: If F is ergodic and $\mathrm{Out}(F)$ is algebraically isomorphic to $\mathrm{Out}(E)$, then F is hyperfinite.

Bezuglyi and Golodets [BG2] have generalized the Connes-Krieger theorem to show that if $\pi_1, \pi_2 : G \to N[E]$ are two homomorphisms of a countable amenable group into $N[E]$ and $\overline{\pi}_i : G \to \mathrm{Out}(E)$ is defined as $\overline{\pi}_i = \varphi \circ \pi_i$, with $\varphi : N[E] \to \mathrm{Out}(E)$ the canonical surjection, then $\overline{\pi}_1, \overline{\pi}_2$ are conjugate (i.e., $\exists \theta \in \mathrm{Out}(E)$ with $\overline{\pi}_1(g) = \theta \overline{\pi}_2(g) \theta^{-1}, \forall g \in G$) iff $\ker(\pi_1) = \ker(\pi_2)$.

Remark. Consider now a Borel measure preserving action of a countable group Γ on (X, μ) and the associated equivalence relation $E_\Gamma^X = E$. We denote by C_Γ the *stabilizer of the action*, i.e., the set of $T \in \mathrm{Aut}(X, \mu)$ such that (writing $\gamma(x) = \gamma \cdot x$) $T\gamma T^{-1} = \gamma, \forall \gamma \in \Gamma$. This is equivalent to $T(\gamma \cdot x) = \gamma \cdot T(x), \forall \gamma \in \Gamma$, i.e., $T \in C_\Gamma$ iff T is an isomorphism of the action. In particular, $C_\Gamma \leq N[E]$. By Proposition 6.2, assuming E is aperiodic, $\tau_{N[E]} | C_\Gamma = w | C_\Gamma$. Thus, since C_Γ is obviously a closed subgroup of $(\mathrm{Aut}(X, \mu), w)$, and therefore Polish, C_Γ is a closed subgroup of $(N[E], \tau_{N[E]})$.

Now $C_\Gamma \leq N[E]$ acts by conjugation on $[E]$. Since $(C_\Gamma, w) = (C_\Gamma, \tau_{N[E]})$ is a closed subgroup of $\mathrm{Iso}([E], \delta_u)$, when we identify $T \in C_\Gamma$ with $S \in [E] \mapsto TST^{-1} \in [E]$, and the evaluation action of $\mathrm{Iso}([E], \delta_u)$ (with the pointwise convergence topology) on $([E], u)$ is continuous, it follows that the conjugation action of (C_Γ, w) on $([E], u)$ is continuous. Consider then the semidirect product $C_\Gamma \ltimes [E]$ (for the conjugation action), which we take here to be the space $C_\Gamma \times [E]$ with the product topology and multiplication defined by

$$(T_1, S_1)(T_2, S_2) = (T_1 T_2, (T_2^{-1} S_1 T_2) S_2)$$

(see Appendix I, **(B)**). This is again a Polish group. Let

$$\varphi : C_\Gamma \ltimes [E] \to N[E]$$

be defined by

$$\varphi(T, S) = TS.$$

This is a continuous homomorphism of the Polish group $C_\Gamma \ltimes E$ into the group $(N[E], \tau_{N[E]})$ whose range is the group

$$C_\Gamma[E]$$

generated by C_Γ and $[E]$. In particular, $C_\Gamma[E]$ is Polishable.

In the case $C_\Gamma \cap [E] = \{1\}$ (and this is quite common, see 14.6 below), then φ is a continuous injective homomorphism of $C_\Gamma \ltimes [E]$ onto $C_\Gamma[E]$. Thus if additionally $C_\Gamma[E]$ is closed, φ is a topological group isomorphism of $C_\Gamma \ltimes [E]$ with $(C_\Gamma[E], \tau_{N[E]})$. Since $\varphi(\{1\} \times [E]) = [E]$, we conclude that $[E]$ is closed in $N[E]$. Thus

$$(C_\Gamma \cap [E] = \{1\} \text{ and } C_\Gamma[E] \text{ is closed in } N[E]) \Rightarrow [E] \text{ is closed in } N[E].$$

Comments. Very interesting explicit calculations of the outer automorphism groups of equivalence relations induced by certain actions have been obtained by Gefter [Ge], Gefter-Golodets [GG], Furman [Fu1], Popa [Po4], Ioana-Peterson-Popa [JPP] and Popa-Vaes [PV]. Among them are examples where $\mathrm{Out}(E)$ is trivial, i.e., $N[E] = [E]$.

8. Costs and the outer automorphism group

(A) We establish here a connection between the cost of an equivalence relation and the structure of its outer automorphism group. Concerning the theory of costs, see Gaboriau [Ga]. We follow in this section the terminology and notation of Kechris-Miller [KM], Ch. 3. In particular, the cost of an equivalence relation E on (X, μ) is denoted by $C_\mu(E)$.

Theorem 8.1 (Kechris). *If the outer automorphism group of an ergodic equivalence relation E is not Polish (i.e., if $[E]$ is not closed in $N[E]$), then $C_\mu(E) = 1$.*

Proof. To say that $[E]$ is not closed in $N[E]$ means that the identity map from $([E], u)$ into $N[E]$ is not a homeomorphism, i.e., the identity map from $([E], \tau_{N[E]})$ into $([E], u)$ is not continuous, or, equivalently, that there is a sequence $\{T_n\} \subseteq [E]$ such that $\delta_u(T_nT, TT_n) \to 0, \forall T \in [E]$, but $\delta_u(T_n, 1) \not\to 0$.

Call a sequence $\{T_n\} \subseteq [E]$ *good* if $\exists \epsilon > 0 \forall n (\delta_u(T_n, 1) \geq \epsilon)$ & $\forall T \in [E](\delta_u(T_nT, TT_n) \to 0)$. Thus $[E]$ is not closed in $N[E]$ iff there is a good sequence. Note also that a subsequence of a good sequence is good.

Lemma 8.2. *If $\{T_n\}$ is good, then $E_{\langle T_n \rangle_n}$(= the equivalence relation induced by $\{T_n\}$) is aperiodic.*

We will assume this and complete the proof.

Lemma 8.3. *If $\{T_n\}$ is good, $S \in [E]$ is aperiodic, and $\epsilon > 0$, then there is a sequence $i_0 < i_1 < i_2 < \dots$ such that $C_\mu(E_{\langle S, T_{i_{k_\ell}} \rangle \ell}) < 1 + \epsilon$ for any subsequence $i_{k_0} < i_{k_1} < i_{k_2} < \dots$.*

Proof. We first show that for any $\delta > 0$ there is $i(\delta)$ such that for any $i \geq i(\delta)$ there is A_i with $\mu(A_i) < \delta$ and $\{S, T_i | A_i\}$ is an L-graphing of $E_{\langle S, T_i \rangle}$. Since S is aperiodic, we can find a complete Borel section A of E_S with $\mu(A) < \delta/3$. Then we can find large N so that $\mu(B) < \delta/3$, where

$$B = \{x \in X : \forall |n| \leq N (S^n(x) \notin A)\}.$$

Next fix $i(\delta)$ such that for all $i \geq i(\delta)$ and $|n| \leq N$,

$$\delta_u(T_iS^n, S^nT_i) = \delta_u(S^{-n}T_iS^n, T_i) < \frac{\delta}{3(2N+1)}.$$

For such i, if

$$C_i = \{x : \exists |n| \leq N (S^{-n}T_iS^n(x) \neq T_i(x))\},$$

then $\mu(C_i) < \delta/3$.

If $x \notin C_i, x \notin B$, there is $|n| \leq N$ with $S^n(x) \in A$ and $T_i(x) = S^{-n}T_iS^n(x)$. Thus $\{S, T_i|(A \cup B \cup C_i)\}$ forms an L-graphing of $E_{\langle S,T_i\rangle}$ and if $A_i = A \cup B \cup C_i$, then $\mu(A_i) < \delta$.

It follows that we can find $i_0 < i_1 < \cdots < i_k < i_{k+1} < \dots$ such that

$$i \geq i_k \Rightarrow \exists B_i(\mu(B_i) < \frac{\epsilon}{2^{k+1}} \text{ and } \{S, T_i|B_i\} \text{ is an L-graphing of } E_{\langle S,T_i\rangle}).$$

Let $i_{k_0} < i_{k_1} < \dots$ be a subsequence of $\{i_k\}$. Then $i_{k_\ell} \geq i_\ell$, so we can find D_ℓ with $\mu(D_\ell) < \frac{\epsilon}{2^{\ell+1}}$ and $\{S, T_{i_{k_\ell}}|D_\ell\}$ is an L-graphing of $E_{\langle S,T_{i_{k_\ell}}\rangle}$, so that $\{S, T_{i_{k_\ell}}|D_\ell\}_\ell$ is an L-graphing of $E_{\langle S,T_{i_{k_\ell}}\rangle\ell}$ and therefore $C_\mu(E_{\langle S,T_{i_{k_\ell}}\rangle\ell}) < 1 + \sum_{\ell=0}^{\infty} \frac{\epsilon}{2^{\ell+1}} = 1 + \epsilon$. □

Lemma 8.4. *If $[E]$ is not closed in $N[E]$ and there are aperiodic transformations $S_1, \dots, S_k \in [E]$ with $E = E_{\langle S_1,\dots,S_k\rangle}$, then $C_\mu(E) = 1$.*

Proof. Fix $\epsilon > 0$. Repeatedly applying the preceding lemma, we can find a good sequence $\{T_n\}$ such that

$$C_\mu(E_{\langle S_i,T_n\rangle_n}) \leq 1 + \frac{\epsilon}{k}, \forall i \leq k.$$

Since, by Lemma 8.2, $E_{\langle T_n\rangle_n}$ is aperiodic, it follows (see Kechris-Miller [KM], 23.5) that

$$\begin{aligned} C_\mu(E) - 1 = \ & C_\mu(\bigvee_{i=1}^{k} E_{\langle S_i,T_n\rangle_n}) - 1 \\ \leq \ & \sum_{i=1}^{k}[C_\mu(E_{\langle S_i,T_n\rangle_n}) - 1] \leq \ \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $C_\mu(E) = 1$. □

The next fact is true in the pure Borel category (where we interpret $[E]$ as consisting of all Borel automorphisms T with $T(x)Ex$, for *all* x).

Lemma 8.5. *Let E be a countable aperiodic Borel equivalence relation and $S \in [E]$. Then there is an aperiodic $T \in [E]$ with $E_S \subseteq E_T$.*

Proof. We can assume that every orbit of S is finite. Let then A be a Borel transversal for E_S. Then $E|A$ is aperiodic, so (by the proof of 3.5, (i)) let $F \subseteq E|A$ be aperiodic hyperfinite. Then $F \vee E_S$ is aperiodic, hyperfinite and so of the form E_T for some aperiodic $T \in [E]$. □

Now, to complete the proof, assume E is as in the statement of the theorem. By 3.5 there is $U_0 \in [E]$ which is ergodic (and thus aperiodic). By 8.3, there is a good sequence $\{T_n\}$ such that $C_\mu(E_{\langle U_0,T_n\rangle_n}) < 2$. Let $E = E_{\langle U_1,U_2,\dots\rangle}, U_i \in [E]$. Then if $E_m = E_{\langle U_0,\dots,U_m,T_n\rangle_n}, E_1 \subseteq E_2 \subseteq \dots$ and each E_m is ergodic. Also

$$C_\mu(E_m) \leq m + C_\mu(E_{\langle U_0,T_n\rangle_n}) < m + 2 < \infty,$$

so (see Kechris-Miller [KM], 27.7) $E_m = E_{\langle S_1,\dots,S_k\rangle}$ for some $S_1,\dots,S_k \in [E_m]$, which by 8.5 we can assume are aperiodic. But $[E_m]$ is not closed in $N[E_m]$ since $\{T_n\}$ is good for E_m as well. So, by 8.4, $C_\mu(E_m) = 1$. As $\bigcup_m E_m = E$, it follows (see Kechris-Miller [KM], 23.5) that $C_\mu(E) = 1$.

So it only remains to give the proof of 8.2.

Proof of Lemma 8.2. I will give an argument due to Ben Miller which is simpler than my original one.

Assume, towards a contradiction, that there is a set of positive measure on which $F = E_{\langle T_n\rangle_n}$ is periodic. It follows that there is a partition $X = A \cup B$ into two F-invariant Borel sets of positive measure with $F|A$ periodic. By ergodicity of E, we can assume that A, B are complete sections for E.

Let $\{g_n\}$ be a sequence of Borel involutions such that $E = \bigcup_n \mathrm{graph}(g_n)$. Also let $\{A_n\}$ be a sequence of partial Borel F-transversals (i.e., each A_n meets every F-class in at most one point) such that $\bigcup_n A_n = A$. Let finally G_n, H_n be pairwise disjoint such that $G_n \cup H_n = \mathrm{supp}(g_n)$ and $g_n(G_n) = H_n$. Put

$$G'_{n,m} = \{x \in G_n \cap A_m : g_n(x) \in B\}, H'_{n,m} = G'_{n,m} \cup g_n(G'_{n,m})$$

and let

$$h_{n,m} = g_n|H'_{n,m} \cup \mathrm{id}|(X \setminus H'_{n,m}).$$

Then $\{h_{n,m}\}$ is a family of involutions and

$$xEy \;\&\; x \in A \;\&\; y \in B \Leftrightarrow \exists n, m(h_{n,m}(x) = y \;\&\; x \neq y).$$

Also $h_{n,m} \in [E]$ and $\bigcup_{n,m} \mathrm{supp}(h_{n,m}) = X$.

Claim. *If* $z \in \mathrm{supp}(h_{n,m})$ *and* $T_k h_{n,m} T_k^{-1}(z) = h_{n,m}(z)$, *then* $T_k(z) = z$.

Proof of the claim. Assume $z \in \mathrm{supp}(h_{n,m})$ and $T_k h_{n,m} T_k^{-1}(z) = h_{n,m}(z)$ but $T_k(z) \neq z$ (and thus $T_k^{-1}(z) \neq z$ as well).

Case (1). $z \in A$. Either $T_k^{-1}(z) \notin \mathrm{supp}(h_{n,m})$, so $T_k h_{n,m} T_k^{-1}(z) = z = h_{n,m}(z)$, which is impossible or $T_k^{-1}(z) \in \mathrm{supp}(h_{n,m})$, so $z \neq T_k^{-1}(z)$ are F-equivalent and in $\mathrm{supp}(h_{n,m}) \cap A \subseteq A_m$, which is an F-transversal, a contradiction.

Case (2). $z \in B$. Let $h_{n,m}(z) = y$, so $y \in A$ and $h_{n,m}(y) = z$. As in Case (1), $T_k^{-1}(z) \in \mathrm{supp}(h_{n,m})$. Put $h_{n,m}(T_k^{-1}(z)) = w$. Then wFy and $w \neq y$. But also $w, y \in \mathrm{supp}(h_{n,m}) \cap A \subseteq A_m$, a contradiction.

So we see that

$$\mathrm{supp}(h_{n,m}) \cap \{z : T_k(z) \neq z\} \subseteq \{z : T_k h_{n,m} T_k^{-1}(z) \neq h_{n,m}(z)\}.$$

Let $A_{n,m} \subseteq \mathrm{supp}(h_{n,m})$ be pairwise disjoint with

$$\bigcup_{n,m} A_{n,m} = \bigcup_{n,m} \mathrm{supp}(h_{n,m}) = X.$$

Then

$$\{z \in A_{n,m} : T_k(z) \neq z\} \subseteq \{z \in A_{n,m} : T_k h_{n,m} T_k^{-1}(z) \neq h_{n,m}(z)\},$$

thus

$$\delta_u(T_k, 1) \leq \sum_{n,m} \mu(A_{n,m} \cap \{z : T_k h_{n,m} T_k^{-1}(z) \neq h_{n,m}(z)\}).$$

Fix $\epsilon > 0$ and find a finite set $F \subseteq \mathbb{N}^2$ such that

$$\sum_{(n,m)\notin F} \mu(A_{n,m}) < \epsilon$$

and then K_0 large enough so that for all $(n, m) \in F$,

$$k \geq K_0 \Rightarrow \mu(\{z : T_k h_{n,m} T_k^{-1}(z) \neq h_{n,m}(z)\}) < \frac{\epsilon}{\text{card}(F)}.$$

Then for such k,

$$\delta_u(T_k, 1) \leq \epsilon + \text{card}(F) \cdot \frac{\epsilon}{\text{card}(F)} = 2\epsilon.$$

Thus $\delta_u(T_k, 1) \to 0$, a contradiction. □

Remark. Note that if $\delta_u(T_n, 1) \to 0$, then there is a subsequence $\{T_{n_i}\}$ such that $E_{\langle T_{n_i}\rangle_i}$ is not aperiodic (in fact it is equality on a set of positive measure). To see this, find $\{n_i\}$ so that $\sum_i \delta_u(T_{n_i}, 1) < \infty$. Then if $B_{n_i} = \text{supp}(T_{n_i}), \sum_i \mu(B_{n_i}) < \infty$, so, by Borel-Cantelli, $\mu(\{x : \forall i \exists j \geq i(x \in B_{n_j})\}) = 0$, or (neglecting as usual null sets)

$$\forall x \exists i \forall j \geq i(T_{n_j}(x) = x).$$

Then for some set B of positive measure and some i,

$$x \in B \Rightarrow \forall j \geq i(T_{n_j}(x) = x),$$

i.e., $E_{\langle T_{n_j}\rangle_{j\geq i}}|B =$ equality$|B$.

Thus we see that, given any sequence $\{T_n\} \subseteq E$ with $T_n \to 1$ in $N[E]$, the following are equivalent:

(i) $\exists \epsilon > 0 \forall n(\delta_u(T_n, 1) \geq \epsilon)$ (i.e., $\{T_n\}$ is good),

(ii) For every subsequence $\{n_i\}, E_{\langle T_{n_i}\rangle i}$ is aperiodic.

Remark. By definition, $[E]$ not closed in $N[E]$ means that there is a sequence $\{T_n\} \subseteq [E]$ with $\delta_u(T_n, 1) \not\to 0$ but $T_n \to 1$ in $N[E]$. We note here that if E is actually E_0-ergodic, then, if $[E]$ is not closed in $N[E]$, there is such $\{T_n\}$ with $\delta_u(T_n, 1) = 1$, i.e., $\text{supp}(T_n) = X$.

Indeed we have some $\{S_n\} \subseteq [E]$ and $\epsilon > 0$ with $\delta_u(S_n, 1) \geq \epsilon$ and $S_n \to 1$ in $N[E]$. Let $A_n = \{x : S_n(x) = x\}$, so that $\mu(X \setminus A_n) \geq \epsilon$.

If $\delta_u(S_n, 1) \not\to 1$, then we can assume that for some $\delta > 0, \delta_u(S_n, 1) \leq 1 - \delta$, so $\mu(A_n) \geq \delta$. Thus $\mu(A_n)(1 - \mu(A_n)) \not\to 0$.

Fix $T \in [E]$. If

$$B_n = \{x : T^{-1}S_n(x) \neq S_n T^{-1}(x)\},$$

then $\mu(B_n) \to 0$ (as $S_n \to 1$ in $N[E]$). If $x \notin B_n$, then

$$\begin{aligned} x \in A_n \Rightarrow \ & T^{-1}S_n(x) = T^{-1}(x) = S_n(T^{-1}(x)) \\ \Rightarrow \ & T^{-1}(x) \in A_n \Rightarrow x \in T(A_n), \end{aligned}$$

so $A_n \backslash T(A_n) \subseteq B_n$ and therefore $\mu(A_n \backslash T(A_n)) \to 0$. Thus $\mu(A_n \Delta T(A_n)) = 2\mu(A_n \setminus T(A_n)) \to 0$. It follows that E is not E_0-ergodic, a contradiction.

Thus $\delta_u(S_n, 1) \to 1$. Let $U_n \in [E]$ be an involution with support A_n and put

$$T_n(x) = \begin{cases} S_n(x), & \text{if } x \notin A_n, \\ U_n(x), & \text{if } x \in A_n. \end{cases}$$

Then $T_n \in [E], \delta_u(S_n, T_n) \leq \mu(A_n) \to 0$ and $\delta_u(T_n, 1) = 1$. Also clearly $T_n \to 1$ in $N[E]$.

Remark. The converse of 8.1 is not true. By 9, **(A)** and 9.1 below there are groups Γ such that for any measure preserving, free, ergodic action of Γ, we have $[E_\Gamma^X]$ closed in $N[E_\Gamma^X]$ but $C_\mu(E_\Gamma^X) = 1$. For example, one can take any ICC (*infinite conjugacy classes*) group with property (T), fixed price and cost equal to 1, for instance $\mathrm{SL}_3(\mathbb{Z})$.

(B) The preceding result 8.1 can be used to give a partial answer to a question of Jones-Schmidt [JS], p. 113 and Schmidt [Sc2], 4.6. They ask for a characterization of when $[E]$ is closed in $N[E]$. The following corollary resolves this when E is *treeable*, i.e., there is a Borel acyclic graph whose connected components are the E-classes.

Corollary 8.6. *Let E be an ergodic equivalence relation. Then if E is treeable, the following are equivalent:*

(i) $[E]$ is closed in $N[E]$.

(ii) E is not hyperfinite.

Proof. We have already seen that if E is hyperfinite, $[E]$ is dense in (and not equal to) $N[E]$. Conversely, if $[E]$ is not closed, then by 8.1, $C_\mu(E) = 1$, so, as E is treeable, E is hyperfinite (see Kechris-Miller [KM], 22.2 and 27.10). □

It is interesting to note here that although the statement of this result does not involve costs, the proof given here makes use of this concept.

For further results concerning the question of when $[E]$ is closed in $N[E]$, see the next section and Section 29, **(C)**.

9. Inner amenability

(A) Recall that a countable group Γ is *inner amenable* if there is a mean on $\Gamma \setminus \{1\}$ invariant under the conjugacy action of Γ or equivalently there is a finitely additive probability measure (f.a.m.) on $\Gamma \setminus \{1\}$ invariant under the conjugacy action of Γ. See Effros [Ef1], where this notion was introduced, and the survey Bedos-de la Harpe [BdlH]. Examples of such groups are:

(i) amenable groups,

(ii) groups that have a finite conjugacy class $\neq \{1\}$,

(iii) direct products $\Gamma \times \Delta$, where $\Gamma \neq \{1\}$ is amenable,

(iv) *weakly commutative groups* (i.e., groups Γ such that for every finite $F \subseteq \Gamma$ there is a $\gamma \in \Gamma \setminus \{1\}$ commuting with each element of F). For example, direct sums $\Gamma_1 \oplus \Gamma_2 \oplus \dots$ of countable non-trivial groups,

(v) the Thompson group $F = \langle x_0, x_1, x_2, \dots | x_i^{-1} x_n x_i = x_{n+1}, i < n \rangle$.

On the other hand the free groups $F_n, n \geq 2$, are not inner amenable. Also ICC groups with property (T) are not inner amenable.

(B) The following result was proved in Jones-Schmidt [JS].

Theorem 9.1 (Jones-Schmidt). *Given a free, measure preserving action of a countable group Γ on (X, μ), if $[E_\Gamma^X]$ is not closed in $N[E_\Gamma^X]$, then Γ is inner amenable.*

Proof. Since, letting $E = E_\Gamma^X, [E]$ is not closed in $N[E]$, there is a sequence $T_n \in [E]$ with $\delta_u(T_n, 1) \geq \epsilon$, some $\epsilon > 0$, and $T_n \gamma T_n^{-1} \to \gamma$ uniformly, $\forall \gamma \in \Gamma$.

Put

$$T_n(x) = \alpha(n, x) \cdot x,$$

where $\alpha(n, x) \in \Gamma$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and define a positive linear functional on $\ell^\infty(\Gamma)$ by

$$\varphi(f) = \lim_{n \to \mathcal{U}} \int f(\alpha(n, x)) d\mu(x).$$

Note that if χ_1 is the characteristic function of $\{1\}$, then $\varphi(\chi_1) \neq 1$ as

$$\begin{aligned} \varphi(\chi_1) &= \lim_{n \to \mathcal{U}} \int \chi_1(\alpha(n, x)) d\mu(x) \\ &= \lim_{n \to \mathcal{U}} \int_{\{x : T_n(x) = x\}} d\mu(x) \\ &= \lim_{n \to \mathcal{U}} \mu(\{x : T_n(x) = x\}) \\ &= \lim_{n \to \mathcal{U}} (1 - \delta_u(T_n, 1)) \leq 1 - \epsilon < 1. \end{aligned}$$

We will next see that φ is conjugacy invariant. Then clearly

$$\psi(f) = \frac{\varphi(f^*)}{1 - \varphi(\chi_1)},$$

where $f^*(\gamma) = f(\gamma)$ if $\gamma \neq 1, f^*(1) = 0$, is a conjugacy invariant mean on $\Gamma \setminus \{1\}$, so we are done.

For $\gamma \in \Gamma, f \in \ell^\infty(\Gamma)$, put $(\gamma \cdot f)(\delta) = f(\gamma^{-1} \delta \gamma)$. We need to verify that $\varphi(\gamma \cdot f) = \varphi(f)$. We have

$$\begin{aligned} \varphi(\gamma \cdot f) &= \lim_{n \to \mathcal{U}} \int (\gamma \cdot f)(\alpha(n, x)) d\mu(x) \\ &= \lim_{n \to \mathcal{U}} \int f(\gamma^{-1} \alpha(n, x) \gamma) d\mu(x). \end{aligned}$$

Let $A_n = \{x : T_n(\gamma^{-1} \cdot x)) \neq \gamma^{-1} \cdot T_n(x)\}$, so that $\mu(A_n) \to 0$. If $x \notin A_n$, then

$$T_n(\gamma^{-1} \cdot x) = \gamma^{-1} \cdot T_n(x),$$

so

$$\alpha(n,\gamma^{-1}\cdot x)\gamma^{-1}\cdot x=\gamma^{-1}\alpha(n,x)\cdot x$$

and, as the action is free,

$$\alpha(n,\gamma^{-1}\cdot x)\gamma^{-1}=\gamma^{-1}\alpha(n,x),$$

so

$$\gamma^{-1}\alpha(n,x)\gamma=\alpha(n,\gamma^{-1}\cdot x).$$

Thus

$$\varphi(\gamma\cdot f)=\lim_{n\to\mathcal{U}}\int_{A_n}f(\gamma^{-1}\alpha(n,x)\gamma)d\mu(x)+\lim_{n\to\mathcal{U}}\int_{X\setminus A_n}f(\alpha(n,\gamma^{-1}\cdot x))d\mu(x).$$

The second summand is equal to

$$\lim_{n\to\mathcal{U}}\int_{\gamma^{-1}\cdot(X\setminus A_n)}f(\alpha(n,x))d\mu(x)=\varphi(f)-\lim_{n\to\mathcal{U}}\int_{\gamma^{-1}\cdot A_n}f(\alpha(n,x))d\mu(x).$$

Thus

$$\begin{aligned}\varphi(\gamma\cdot f)-\varphi(f)\ &=\lim_{n\to\mathcal{U}}\int_{A_n}f(\gamma^{-1}\alpha(n,x)\gamma)d\mu(x)-\\ &\lim_{n\to\mathcal{U}}\int_{\gamma^{-1}\cdot A_n}f(\alpha(n,x))d\mu(x),\end{aligned}$$

which equals 0, since $\mu(A_n)\to 0$. □

It follows from 9.1 that for every free, measure preserving action of a countable group Γ, which is ICC and has property (T), $[E_\Gamma^X]$ is closed in $N[E_\Gamma^X]$ (see Gefter-Golodets [GG]).

(C) The converse of 9.1 is literally false as stated, in view of the following simple fact.

Proposition 9.2. *If $\Gamma=\Delta\times F_2$, where Δ is non-trivial finite, then for every free, measure preserving, ergodic action on (X,μ), $[E_\Gamma^X]$ is closed in $N[E_\Gamma^X]$.*

Proof. This follows from 8.1, as the cost $C(\Gamma)$ is greater than 1 and Γ has fixed price but one can also give a direct proof.

Let $E=E_\Gamma^X$. Let $Y=X/\Delta$ (we view here Δ as a subgroup of Γ and similarly for F_2). Let $\pi: X\to Y$ be the canonical projection and put $\nu=\pi_*\mu$. Then F_2 acts on Y by $\gamma\cdot(\Delta\cdot x)=\Delta\cdot(\gamma\cdot x)$. This is a free, measure preserving, ergodic action of F_2 on Y. If $[E_\Gamma^X]$ is not closed in $N[E_\Gamma^X]$, we will show that $[E_{F_2}^Y]$ is not closed in $N[E_{F_2}^Y]$, contradicting 9.1.

Let $T_n\in[E_\Gamma^X]$ be such that $\delta_u(T_n,1)\not\to 0$ but $T_n\to 1$ in $N[E_\Gamma^X]$. Thus if $A_n=\{x:\forall\delta\in\Delta(T_n(\delta\cdot x)=\delta\cdot T_n(x))\}$, then $\mu(A_n)\to 1$. Then for every $\Delta\cdot x\in Y$ such that $\Delta\cdot x\cap A_n\neq\emptyset$ define

$$\tilde{T}_n(\Delta\cdot x)=\Delta\cdot T_n(y),\ \text{for}\ y\in\Delta\cdot x\cap A_n.$$

This is well-defined. Clearly $\tilde{T}_n(\Delta \cdot x) E^Y_{F_2} \Delta \cdot x$ and $\tilde{T}_n$ is 1-1 and measure preserving on its domain, which has ν-measure equal to $\mu(\Delta \cdot A_n) \to 1$. So we can extend $\tilde{T}_n$ to $S_n \in [E^Y_{F_2}]$.

We will check that $\{S_n\}$ witnesses that $[E^Y_{F_2}]$ is not closed in $N[E^Y_{F_2}]$. For this it is enough to verify:

(i) $S_n \to 1$ weakly,

(ii) $S_n \gamma S_n^{-1} \to \gamma$ uniformly, $\forall \gamma \in F_2$,

(iii) $\delta_u(S_n, 1) \not\to 0$.

Now (i), (ii) easily follow from the fact that $\{T_n\}$ has the same properties. To prove (iii) we will use the following simple fact: If E is a smooth equivalence relation and $U_n \in [E], U_n \to 1$ weakly, then $U_n \to 1$ uniformly. This is because w and u agree on $[E]$, as $[E]$ is weakly closed (or write $X = \bigcup_i A_i$, where $\{A_i\}$ is a pairwise disjoint family of partial Borel transversals and note that $\mu(\{x \in A_i : U_n(x) \neq x\}) = \frac{1}{2}\mu(U_n(A_i)\Delta A_i))$. Assume then that (iii) fails, towards a contradiction. From the definition it follows then that there is $U_n \in [E^X_\Delta]$ such that $\delta_u(U_n, T_n) \to 0$ and so $U_n \to 1$ weakly (since $T_n \to 1$ weakly) and $U_n \not\to 1$ uniformly, a contradiction. □

Schmidt [Sc2] raised the question of whether there is a converse of 9.1 for ICC groups.

Problem 9.3. *Assume that* Γ *is countable,* ICC *and inner amenable. Is there a free, measure preserving, ergodic action of* Γ *for which* $[E^X_\Gamma]$ *is not closed in* $N[E^X_\Gamma]$*?*

In connection with this question we note the following corollary of 8.1, 8.6.

Theorem 9.4 (Kechris). *Suppose* Γ *is a countable group that admits a free, measure preserving, ergodic action on* (X, μ) *with* $[E^X_\Gamma]$ *not closed in* $N[E^X_\Gamma]$*. Then* $C(\Gamma) = 1$*. If moreover,* Γ *is* strongly treeable (*i.e., every equivalence relation induced by a free, measure preserving action of* Γ *is treeable*)*, then* Γ *is amenable. Thus for strongly treeable groups* Γ*,* Γ *is amenable iff there is a free, measure preserving, ergodic action of* Γ *on* (X, μ)*, with* $[E^X_\Gamma]$ *not closed in* $N[E^X_\Gamma]$*.*

Note that, in view of 9.4, a positive answer to 9.3 implies that $C(\Gamma) = 1$ for any ICC inner amenable group, which is also an open problem.

Concerning 9.3, we also have the following partial answer.

Proposition 9.5. *Let* Γ *be a countable* ICC *group which is weakly commutative. Then there is a free, measure preserving, ergodic action of* Γ *with* $[E^X_\Gamma]$ *not closed in* $N[E^X_\Gamma]$*.*

Before we start the proof, let us notice certain general facts about conjugacy shift actions. Let Γ be a countable group, put

$$\Gamma^* = \Gamma \setminus \{1\},$$

and consider its *conjugacy shift action* on $2^{\Gamma^*} : (\gamma \cdot p)\delta = p(\gamma^{-1}\delta\gamma)$.

Proposition 9.6. *Let* Γ *be an infinite countable group. Then* Γ *is* ICC *iff its conjugacy shift action on* 2^{Γ^*} *is ergodic iff its conjugacy shift action on* 2^{Γ^*} *is weak mixing.*

Proof. View 2^{Γ^*} as $\mathbb{Z}_2^{\Gamma^*}$ = the Cantor group. The conjugacy shift action of Γ on $\mathbb{Z}_2^{\Gamma^*}$ is an action by automorphisms of $\mathbb{Z}_2^{\Gamma^*}$ and the corresponding action on $\widehat{\mathbb{Z}_2^{\Gamma^*}} = \mathbb{Z}_2^{<\Gamma^*} = \mathcal{P}_{\text{fin}}(\Gamma^*)$ (= the set of finite subsets of Γ^*) is the conjugacy action of Γ on $\mathcal{P}_{\text{fin}}(\Gamma^*)$. Let $\mathcal{P}^*_{\text{fin}}(\Gamma) = \mathcal{P}_{\text{fin}}(\Gamma^*)\setminus\{\emptyset\}$ and denote by Γ_F the stabilizer of $F \in \mathcal{P}^*_{\text{fin}}(\Gamma)$ under this action. Since the trivial character corresponds to $\emptyset$, we have that the conjugacy shift action is ergodic iff it is weak mixing iff $\forall F \in \mathcal{P}^*_{\text{fin}}(\Gamma)([\Gamma : \Gamma_F] = \infty)$ (see, e.g., Kechris [Kec4], 4.3).

If $\gamma \in \Gamma^*$, then $\Gamma_{\{\gamma\}}$ has infinite index iff the conjugacy class of γ is infinite. Also $[\Gamma_F : \Gamma_F \cap \Gamma_{\{\gamma\}}] < \infty$ for any $\gamma \in F$, thus $\forall F \in \mathcal{P}^*_{\text{fin}}(\Gamma)([\Gamma : \Gamma_F] = \infty)$ iff $\forall \gamma \in \Gamma^*([\Gamma : \Gamma_{\{\gamma\}}] = \infty)$ iff Γ is ICC. □

Also note the following fact.

Proposition 9.7. *If* Γ *is countable* ICC, *then the conjugacy shift action of* Γ *on* 2^{Γ^*} *is free (a.e.).*

Proof. Fix $\gamma \in \Gamma^*$ and let $p \in 2^{\Gamma^*}$ be such that $\gamma \cdot p = p$, i.e., $\forall\delta(p(\delta) = p(\gamma^{-1}\delta\gamma))$. Thus if $\langle\gamma\rangle$ acts by conjugation on Γ^*, p is constant on each orbit. Thus $\{p : \gamma\cdot p = p\}$ is null unless all orbits of this action are finite and all but finitely many are trivial, i.e., γ commutes with all but finitely many elements of Γ^* in which case its conjugacy class is finite, a contradiction. □

Proof of 9.5. Let $X = 2^{\Gamma^*}$, μ = the usual product measure and let Γ act on 2^{Γ^*} by conjugacy shift. This action is free, measure preserving and ergodic. Let $\{\gamma_n\} \subseteq \Gamma^*$ *witness the weak commutativity* of Γ, i.e., for each $\delta \in \Gamma, \delta\gamma_n = \gamma_n\delta$ for all large enough n. View each $\gamma \in \Gamma$ as an element of $[E^X_\Gamma]$.

(i) $\gamma_n \to 1$ weakly: It is enough to show for each set A of the form $A = \{p \in 2^{\Gamma^*} : p(\delta_i) = a_i, i = 1,\ldots,k\}$, where $a_i \in \{0,1\}, \delta_1,\ldots,\delta_k \in \Gamma^*$, that $\mu(\gamma_n(A)\Delta A) \to 0$. Now for all large enough n:

$$\begin{aligned} p \in \gamma_n(A) &\Leftrightarrow \gamma_n^{-1}\cdot p \in A \\ &\Leftrightarrow \forall i \le k((\gamma_n^{-1}\cdot p)(\delta_i) = a_i) \\ &\Leftrightarrow \forall i \le k(p(\gamma_n\delta_i\gamma_n^{-1}) = a_i) \\ &\Leftrightarrow \forall i \le k(p(\delta_i) = a_i) \\ &\Leftrightarrow p \in A. \end{aligned}$$

(ii) $\gamma_n\delta\gamma_n^{-1} \to \delta$ uniformly, $\forall\delta \in \Gamma$: This is clear as $\gamma_n\delta\gamma_n^{-1} = \delta$, if n is large enough.

(iii) $\delta_u(\gamma_n, 1) = 1$, as Γ acts freely. □

(D) Recall from Jones-Schmidt [JS] that a measure preserving action of Γ on (X,μ) is called *stable* if E^X_Γ (on (X,μ)) is isomorphic to $E^X_\Gamma \times E_0$ (on $(X,\mu)\times(2^{\mathbb{N}},\nu)$, where ν is the usual product measure on $2^{\mathbb{N}}$). In

Theorem 3.4 of Jones-Schmidt [JS] this is characterized by the existence of a sequence $\{T_n\} \subseteq [E_\Gamma^X]$ for which we have $T_n \to 1$ in $N[E_\Gamma^X]$ but for some non-trivial asymptotically invariant sequence $\{A_n\} \subseteq \mathrm{MALG}_\mu$ we have $\mu(T_n(A_n)\Delta A_n) \not\to 0$. (Here $\{A_n\}$ is a *non-trivial asymptotically invariant sequence* if $\mu(\gamma \cdot A_n \Delta A_n) \to 0, \forall \gamma \in \Gamma$, but $\mu(A_n)(1-\mu(A_n)) \not\to 0$.) Note that for such $\{T_n\}, T_n \not\to 1$ uniformly, so $\{T_n\}$ is a witness to the fact that $[E_\Gamma^X]$ is not closed in $N[E_\Gamma^X]$.

Proposition 9.8. *Suppose there exist* $\{\gamma_n\}, \{\delta_n\} \subseteq \Gamma$ *that witness weak commutativity of* Γ *but* $\gamma_n\delta_n \neq \delta_n\gamma_n, \forall n$. (*For example, let* $\Gamma = \bigoplus_n \Gamma_n, \Gamma_n$ *non-abelian.*) *Then the conjugacy shift action of* Γ *on* 2^{Γ^*} *is stable.*

Proof. As in the proof of 9.5, $\gamma_n \to 1$ in $N[E_\Gamma]$. Put $A_n = \{p \in 2^{\Gamma^*} : p(\delta_n) = 1\}$. Then $\gamma_n \cdot A_n = \{p \in 2^{\Gamma^*} : p(\gamma_n\delta_n\gamma_n^{-1}) = 1\}$, and $\gamma_n\delta_n\gamma_n^{-1} \neq \delta_n$, so

$$\mu(\gamma_n \cdot A_n \cap A_n) = \mu(\gamma_n \cdot A_n)\mu(A_n) = \frac{1}{4}.$$

Thus $\mu(\gamma_n \cdot A_n \Delta A_n) = \frac{1}{2}$. Finally notice that $\mu(\gamma \cdot A_n\Delta A_n) \to 0, \forall\gamma \in \Gamma$, as $\gamma \cdot A_n = A_n$, if n is large enough. □

We now have the following application, where two actions are *orbit equivalent* if the induced equivalence relations are isomorphic (see Section 10, **(A)**).

Proposition 9.9. *Let* $\{\Gamma_n\}$ *be a sequence of countable non-trivial groups at least one of which is not amenable. Then the shift action of* $\Gamma = \bigoplus_n \Gamma_n$ *on* 2^Γ *is not orbit equivalent to the conjugacy shift action on* 2^{Γ^*}.

Proof. Since the shift action of Γ on 2^Γ is E_0-ergodic, it is enough to show that the conjugacy shift action is not.

Case 1. Eventually the Γ_n are abelian. Then $\Gamma = \Delta \times Z$, Z infinite abelian. Consider

$$2^\Gamma \ni p \mapsto f(p) \in 2^{Z^*},$$

where $f(p)(z) = p(1,z)$. Then if p, p' are shift conjugate, clearly $f(p) = f(p')$. Thus the conjugacy shift action of Γ on 2^{Γ^*} is not ergodic, as f cannot be constant on a conull set.

Case 2. Infinitely many Γ_n are not abelian, say for $n = n_1 < n_2 < \dots$. Choose $a_{n_i}, b_{n_i} \in \Gamma_{n_i}$ that do not commute. Let $\gamma_i = (1, \dots, a_{n_i}, 1, \dots), \delta_i = (1, \dots, b_{n_i}, 1, \dots)$. These satisfy the hypothesis of 9.8, so the conjugacy shift action of Γ on 2^{Γ^*} is stable, thus not E_0-ergodic. □

Addendum. Recently Tsankov showed that for any inner amenable group Γ the conjugacy shift action of Γ on 2^{Γ^*} is not E_0-ergodic and thus 9.9 is true for any non-amenable, inner amenable group.

Kechris and Tsankov [KT] then studied more generally the action of a countable group Γ on a countable set I and its relationship to the corresponding shift action of Γ on 2^I. They proved that the following are equivalent: i) The action of Γ on I is amenable (i.e., admits a finitely additive probability measure), ii) the shift action of Γ on 2^I admits non-trivial almost invariant sets (equivalently: admits a non-trivial asymptotically invariant sequence), iii) the Koopman representation associated to the shift action, restricted to the orthogonal of the constant functions, admits non-0 almost invariant vectors (see Section 10, (**C**) for an explanation of these notions). In particular, if these conditions hold, the shift action of Γ on 2^I is not orbit equivalent to the shift action of Γ on 2^Γ, provided Γ is not amenable.

We finally note the following (almost) characterization of weakly commutative groups.

Proposition 9.10. *Let Γ be a countable group.*

i) If Γ is a non-discrete (i.e., not closed) normal subgroup of a Polish group, then Γ is weakly commutative.

ii) If Γ is weakly commutative with trivial center, then Γ is a non-discrete normal subgroup of a Polish group.

Proof. i) Assume that Γ is a non-discrete normal subgroup of the Polish group G. Consider the map

$$\pi : G \to \mathrm{Aut}(\Gamma),$$
$$\pi(g) = (\gamma \mapsto g\gamma g^{-1}).$$

This is a Borel homomorphism, so it is continuous, where $\mathrm{Aut}(\Gamma)$ has the pointwise convergence topology. Since Γ is not discrete in G, there is a sequence $\{\gamma_n\} \subseteq \Gamma \setminus \{1\}$ with $\gamma_n \to 1$ (in G). So $\forall\delta\forall^\infty n(\gamma_n\delta\gamma_n^{-1} = \delta)$, i.e., $\forall\delta\forall^\infty n(\gamma_n\delta = \delta\gamma_n)$, where $\forall^\infty$ means "for all but finitely many". Thus Γ is weakly commutative.

ii) Consider the map

$$\rho : \Gamma \to \mathrm{Aut}(\Gamma) = G,$$
$$\rho(\gamma) = (\delta \mapsto \gamma\delta\gamma^{-1}).$$

Since Γ has trivial center, ρ is an isomorphism, so we can identify Γ with $\rho(\Gamma)$, which is a normal subgroup of G. Let $\{\gamma_n\} \subseteq \Gamma \setminus \{1\}$ witness the weak commutativity of Γ. Then $\gamma_n \neq 1, \gamma_n \to 1$ in G, so Γ is not discrete in G. □

Thus a centerless group is weakly commutative iff it is (up to algebraic isomorphism) a non-discrete normal subgroup of a Polish group (or equivalently a dense normal subgroup of a non-discrete Polish group). In the case of groups with non-trivial center (which are automatically weakly commutative) it is not clear if there is a simple characterization of when they are non-discrete normal subgroups of Polish groups. Tsankov pointed out that if Γ has non-trivial finite center and countable automorphism group $\mathrm{Aut}(\Gamma)$, then Γ cannot be a non-discrete normal subgroup of a Polish group G. Otherwise if $\pi : G \to \mathrm{Aut}(\Gamma)$ is defined as in the proof of 9.10, i) and γ_n are distinct in Γ with $\gamma_n \to 1$, then $\pi(\gamma_n) \to 1$ in $\mathrm{Aut}(\Gamma)$, which is discrete, so $\pi(\gamma_n) = 1$ for all large enough n and thus the center of Γ is infinite. For example, one can take $\Gamma = \Delta \times \mathbb{Z}_2$, where Δ is a simple countable group which is perfect, i.e., every automorphism is inner. On the other hand, it is well-known that if Γ is infinite abelian, then Γ is a subgroup of a compact abelian Polish group and thus a non-discrete normal subgroup of a Polish group. Indeed, if Γ is an infinite abelian group and $\hat{\Gamma}$ is its dual, there is a countable subset $I \subseteq \hat{\Gamma}$ which separates points (i.e., $\forall \gamma \in \Gamma \setminus \{1\} \exists \hat{\gamma} \in I(\hat{\gamma}(\gamma) \neq 1)$). Then consider the (compact abelian Polish) product group $G = \mathbb{T}^I$ and embed (algebraically) Γ into G via $\pi(\gamma) = \gamma | I$ (where we view here γ as a character of $\hat{\Gamma}$).

In the reverse direction, it is also not clear what non-discrete Polish groups admit a dense normal subgroup. Clearly every non-discrete abelian Polish group and the infinite symmetric group have this property.

CHAPTER II

The space of actions

10. Basic properties

Convention. *In the sequel, we consider countable (discrete) groups* Γ*, unless otherwise explicitly stated.*

(A) Let Γ be a countable group and (X, μ) a standard probability space. Every measure preserving action a of Γ on (X, μ) gives rise to a homomorphism $\pi : \Gamma \to \text{Aut}(X, \mu)$, where $\pi(\gamma) = \gamma^a$ with $\gamma^a(x) = a(\gamma, x)$. Conversely every homomorphism $\pi : \Gamma \to \text{Aut}(X, \mu)$ gives rise to a Borel action a of Γ on X such that for each $\gamma \in \Gamma, a(\gamma, x) = \pi(\gamma)(x)$, a.e. Then identifying, as usual, two actions a, b if they agree almost everywhere (i.e., $a(\gamma, x) = b(\gamma, x)$, a.e., $\forall \gamma$), we can view the space of measure preserving actions of Γ in (X, μ) as the space of all homomorphisms of Γ into $\text{Aut}(X, \mu)$, denoted by $A(\Gamma, X, \mu)$ or just $A(\Gamma)$ if (X, μ) is understood. If $\text{Aut}(X, \mu)$ is equipped with either the weak or the uniform topology and $\text{Aut}(X, \mu)^\Gamma$ with the product topology (which we also denote by w or u), then $\text{A}(\Gamma, X, \mu)$ is a closed subspace of $\text{Aut}(X, \mu)^\Gamma$, so $(A(\Gamma, X, \mu), w)$ is Polish and $(A(\Gamma, X, \mu), u)$ is completely metrizable. Thus $a_n \to a$ weakly (uniformly) iff $\gamma^{a_n} \to \gamma^a$ weakly (uniformly), $\forall \gamma \in \Gamma$. Moreover, if $\Gamma = \{\gamma_n\}$, then

$$\delta_{\Gamma,w}(a, b) = \sum_n 2^{-n} \bar{\delta}_w(\gamma_n^a, \gamma_n^b)$$

is a complete compatible metric for w and

$$\delta_{\Gamma,u}(a, b) = \sum_n 2^{-n} \delta_u(\gamma_n^a, \gamma_n^b)$$

is a compatible complete metric for u (and similarly for $\delta'_{\Gamma,u}$ defined by the same formula using δ'_u instead of δ_u)

The group $\text{Aut}(X, \mu)$ acts on $A(\Gamma, X, \mu)$ by conjugation: $(T, a) \mapsto TaT^{-1}$, where

$$\gamma^{TaT^{-1}} = T\gamma^a T^{-1}, \forall \gamma \in \Gamma.$$

This is a continuous action in w or u. Note that $\delta_{\Gamma,u}, \delta'_{\Gamma,u}$ are invariant under conjugation.

If a, b are measure preserving actions of Γ on $(X, \mu), (Y, \nu)$, resp., we say that they are *isomorphic*,

$$a \cong b,$$

if there is a measure preserving bijection $\varphi : X \to Y$ such that $\varphi \circ \gamma^a = \gamma^b \circ \varphi, \forall \gamma \in \Gamma$. Thus two actions $a, b \in A(\Gamma, X, \mu)$ are isomorphic iff they are conjugate. We call a, b *orbit equivalent*,

$$a\mathrm{OE}b,$$

if the equivalence relations induced by a, b are isomorphic, i.e., there is a measure preserving bijection $\varphi : X \to Y$ such that if

$$E_a = \{(x, y) : \exists \gamma(a(\gamma, x) = y)\},$$

then $xE_ay \Leftrightarrow \varphi(x)E_b\varphi(y)$ modulo null sets. We also write this as $E_a \cong E_b$. Clearly $a \cong b \Rightarrow a\mathrm{OE}b$.

If $a, b \in A(\Gamma, X, \mu)$, let also

$$a \equiv b \Leftrightarrow E_a = E_b \text{ (modulo null sets).}$$

Then clearly $\mathrm{OE} = (\equiv) \vee (\cong)$ and $\equiv, \cong$ commute, i.e., $a\mathrm{OE}b \Leftrightarrow \exists c(a \equiv c \cong b)$ iff $\exists c(b \equiv c \cong a)$.

Remark. Suppose Γ is given in terms of generators and relations,

$$\Gamma = \langle x_1, x_2, \dots | w_1 = 1, w_2 = 1, \dots \rangle,$$

where $x_1, x_2, \dots$ may be finite or infinite and similarly for $w_1, w_2, \dots$ Then the space of homomorphisms of Γ into $\mathrm{Aut}(X, \mu)$ can be identified, in either w or u, with the closed subspace of $\mathrm{Aut}(X, \mu)^N$, where N is the cardinality of the set of generators, consisting of all $\{T_1, T_2, \dots\} \in \mathrm{Aut}(X, \mu)^N$ such that $w_1^{T_1, T_2 \dots} = w_2^{T_1, T_2 \dots} = \dots = 1$. Note that this subspace is conjugacy invariant and this identification preserves the conjugacy action of $\mathrm{Aut}(X, \mu)$. In particular, the set of actions of F_N, the free group on N generators, can be identified with $\mathrm{Aut}(X, \mu)^N$.

Finally, we denote by $\mathrm{ERG}(\Gamma, X, \mu)$, $\mathrm{WMIX}(\Gamma, X, \mu)$, $\mathrm{MMIX}(\Gamma, X, \mu)$, $\mathrm{MIX}(\Gamma, X, \mu)$, $\mathrm{E_0RG}(\Gamma, X, \mu)$ the sets of ergodic, weak mixing, mild mixing, mixing, E_0-ergodic actions in $A(\Gamma, X, \mu)$. (These sets are of course empty if Γ is finite.) Also $\mathrm{FR}(\Gamma, X, \mu), \mathrm{FRERG}(\Gamma, X, \mu)$ denote, resp., the sets of free, resp., free ergodic actions.

Recall here that an action $a(\gamma, x) = \gamma \cdot x$ of Γ on (X, μ) is *ergodic* if every invariant Borel set has measure 0 or 1. It is *weak mixing* if the product action $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$ on (X^2, μ^2) is ergodic. It is *mild mixing* if for any Borel set $A \subseteq X$ with $0 < \mu(A) < 1, \underline{\lim}_{\gamma \to \infty} \mu(A \Delta \gamma \cdot A)) > 0$. It is *mixing* if $\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B)$, for any two Borel sets A, B. It is E_0-*ergodic* or *strongly ergodic* if E_a is E_0-ergodic. Finally it is *free* if $\forall \gamma \neq 1(\gamma \cdot x \neq x, \mu\text{–a.e.}(x))$. There are many equivalent formulations of these notions that we will use in the sequel, for which we refer the reader to [BGo], [BR], [Gl2], [HK3], [Sc4].

(B) There is also another way to view the space of actions of Γ, studied in Glasner-King [GK] (see also Rudolph [Rud] for the case $\Gamma = \mathbb{Z}$). We will take as our model space (X, μ) the space $(\mathbb{T}, \lambda)$, where $\mathbb{T}$ = unit circle and λ = (normalized) Lebesgue measure on $\mathbb{T}$ (the interval $[0, 1]$ with Lebesgue measure or the space $2^{\mathbb{N}}$ with the usual product measure would work as well). Consider also $\mathbb{T}^\Gamma$ and the shift action s given by $s(\gamma, p)(\delta) = p(\gamma^{-1}\delta)$. This is clearly a continuous action and we denote by $\mathrm{SIM}(\Gamma)$ the space of probability measures on $\mathbb{T}^\Gamma$ which are shift-invariant. In the space of probability Borel measures on $\mathbb{T}$ endowed with the usual weak*-topology, $\mathrm{SIM}(\Gamma)$ is a compact, convex set. For any $\mu \in \mathrm{SIM}(\Gamma)$ and $\gamma \in \Gamma$, let $\pi_\gamma : \mathbb{T}^\Gamma \to \mathbb{T}$ be the corresponding projection $\pi_\gamma(p) = p(\gamma)$, and consider $(\pi_\gamma)_*\mu$. By shift-invariance this is independent of γ and we denote it as μ_1 $(= (\pi_1)_*\mu)$. It is called the *marginal* of μ. We denote by $\mathrm{SIM}_\nu(\Gamma)$ the space of all $\mu \in \mathrm{SIM}(\Gamma)$ with marginal ν. It is again a compact, convex set.

For each $a \in A(\Gamma, \mathbb{T}, \lambda), x \in \mathbb{T}$, let $\varphi^a(x) \in \mathbb{T}^\Gamma$ be the orbit map of x, i.e.,

$$\varphi^a(x)(\gamma) = a(\gamma^{-1}, x).$$

Then $\varphi^a : \mathbb{T} \to \mathbb{T}^\Gamma$ is a measurable injection (with $\varphi^a(x)(1) = x$), and $\varphi^a(a(\gamma, x)) = s(\gamma, \varphi^a(x))$, so $(\varphi^a)_*(\lambda) = \mu_a$ is in $\mathrm{SIM}(\Gamma)$ with marginal λ. Moreover a on $(\mathbb{T}, \lambda)$ is isomorphic to s on $(\mathbb{T}^\Gamma, \mu_a)$. Glasner-King show that $\Phi_\lambda(a) = \mu_a$ is a homeomorphism of $(A(\Gamma, \mathbb{T}, \lambda), w)$ with a dense G_δ subset of $\mathrm{SIM}_\lambda(\Gamma)$.

Similarly, if μ is any non-atomic probability measure on $\mathbb{T}$ with support $\mathbb{T}$, there is a corresponding homeomorphism Φ_μ (defined exactly as before) of $A(\Gamma, \mathbb{T}, \mu)$ with a dense G_δ subset of $\mathrm{SIM}_\mu(\Gamma)$.

Let H be the Polish group of orientation preserving homeomorphisms of $\mathbb{T}$ that fix 1. Then $h \in H \mapsto h_*\lambda$ is a bijection of H with the set of all non-atomic, full support probability measures on $\mathbb{T}$. Moreover, if for $h \in H$ we let $h^\Gamma : \mathbb{T}^\Gamma \to \mathbb{T}^\Gamma$ be defined by $h^\Gamma(p)(\gamma) = h(p(\gamma))$, then the following diagram commutes

$$\begin{array}{ccc} A(\Gamma, \mathbb{T}, \mu) & \xrightarrow{\Phi_\mu} & \mathrm{SIM}_\mu(\Gamma) \\ \uparrow \bar{h} & & \uparrow h^\Gamma \\ A(\Gamma, \mathbb{T}, \lambda) & \xrightarrow{\Phi_\lambda} & \mathrm{SIM}_\lambda(\Gamma) \end{array}$$

where $\bar{h}$ denotes the obvious homeomorphism of $A(\Gamma, \mathbb{T}, \lambda)$ with $A(\Gamma, \mathbb{T}, \mu)$ induced by h, i.e., $\bar{h}(a)(\gamma, h(x)) = h(a(\gamma, x))$. Note that $\bar{h}(a) \cong a$. Now Glasner-King show that the map

$$\Phi : H \times A(\Gamma, \mathbb{T}, \lambda) \to \mathrm{SIM}(\Gamma),$$

defined by

$$\Phi(h, a) = h^\Gamma_*(\Phi_\lambda(a)) = \Phi_{h_*\lambda}(\bar{h}(a))$$

is a homeomorphism of $H \times A(\Gamma, \mathbb{T}, \lambda)$ with a dense G_δ subset of $\mathrm{SIM}(\Gamma)$. For further reference, we denote by $\mathrm{ESIM}(\Gamma)$ the set of ergodic measures in $\mathrm{SIM}(\Gamma)$. They are the extreme points of $\mathrm{SIM}(\Gamma)$ and they form a G_δ set in $\mathrm{SIM}(\Gamma)$.

(C) Motivated by an analogous notion concerning unitary representations (see Appendix H), it is also useful to introduce a notion of weak containment for actions.

Suppose $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$. We say that a is *weakly contained* in b, in symbols

$$a \prec b,$$

if for any $A_1, \dots, A_n \in \mathrm{MALG}_\mu, F \subseteq \Gamma$ finite, $\epsilon > 0$, there are $B_1, \dots, B_n \in \mathrm{MALG}_\nu$ such that $|\mu(\gamma^a(A_i) \cap A_j) - \nu(\gamma^b(B_i) \cap B_j)| < \epsilon, \forall \gamma \in F, i, j \leq n$. It is clear that we can restrict here the $A_1, \dots, A_n$ to belong to any countable dense subalgebra of MALG_μ and also form a partition of X. We also say that a, b are *weakly equivalent*, in symbols $a \sim b$, if $a \prec b$ and $b \prec a$. We will now provide an equivalent characterization of weak containment (analogous to that in H.2).

Proposition 10.1. *Let $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$. Then*

$$a \prec b \Leftrightarrow a \in \overline{\{c \in A(\Gamma, X, \mu) : c \cong b\}},$$

where closure is in the weak topology.

Proof. $\Leftarrow$: Fix $A_1, \dots, A_n \in \mathrm{MALG}_\mu, F \subseteq \Gamma$ finite, $\epsilon > 0$. Then there is $c \cong b$ such that $\forall \gamma \in F \forall i, j \leq n(|\mu(\gamma^a(A_i) \cap A_j) - \mu(\gamma^c(A_i) \cap A_j)| < \epsilon)$. If $\varphi : (X, \mu) \to (Y, \nu)$ is a Borel isomorphism that sends c to b, put $\varphi(A_i) = B_i$. Then $\varphi(\gamma^c(A_i) \cap A_j) = \gamma^b(B_i) \cap B_j$, so clearly $\forall \gamma \in F \forall i, j \leq n(|\mu(\gamma^a(A_i) \cap A_j) - \nu(\gamma^b(B_i) \cap B_j)| < \epsilon)$.

$\Rightarrow$: By Section 1, **(B)**, basic open nbhds of a in $(A(\Gamma, X, \mu), w)$ have the form

$$V = \{c : \forall \gamma \in F \forall i, j \leq n(|\mu(\gamma^a(A_i) \cap A_j) - \mu(\gamma^c(A_i) \cap A_j)| < \epsilon)\},$$

where $A_1, \dots, A_n \in \mathrm{MALG}_\mu$ form a partition of $X, \epsilon > 0$ and $F \subseteq \Gamma$ is finite containing $1 \in \Gamma$. We need to find $c \cong b$ in any such nbhd. Since $a \prec b$, for any $\delta > 0$ we can find $B_1, \dots, B_n$ in MALG_ν such that

$$\forall \gamma \in F \forall i, j \leq n(|\mu(\gamma^a(A_i) \cap A_j) - \nu(\gamma^b(B_i) \cap B_j)| < \delta).$$

Take $\delta < \frac{\epsilon}{20n^3}$. Since $1 \in F$, we have

$$|\mu(A_i \cap A_j) - \nu(B_i \cap B_j)| < \delta, \forall i, j \leq n,$$

so $\nu(B_i \cap B_j) < \delta$, if $i \neq j \leq n$, and $|\mu(A_i) - \nu(B_i)| < \delta$, so $|1 - \sum_{i=1}^n \nu(B_i)| < n\delta$. Let $B'_i = B_i \setminus \bigcup_{j<i} B_j = \bigcap_{j<i}(B_i \setminus B_j)$. Then $B'_1, \dots, B'_n$ are pairwise disjoint, $\nu(B_i \Delta B'_i) = \nu(B_i \setminus B'_i) = \nu(\bigcup_{j<i}(B_i \cap B_j)) \leq i\delta \leq n\delta$, and $|1 - \sum_{i=1}^n \nu(B'_i)| \leq n^3\delta$, therefore we can find $\bar{B}_1, \dots, \bar{B}_n \in \mathrm{MALG}_\nu$, a partition of Y, with $\nu(B_i \Delta \bar{B}_i) \leq 2n^3\delta$, from which it follows that $|\nu(\gamma^b(B_i) \cap B_j) - \nu(\gamma^b(\bar{B}_i) \cap \bar{B}_j)| \leq 4n^3\delta$ and so $|\mu(\gamma^a(A_i) \cap A_j) - \nu(\gamma^b(\bar{B}_i) \cap \bar{B}_j)| < 5n^3\delta, \forall \gamma \in F, \forall i, j \leq n$. In particular, $|\mu(A_i) - \nu(\bar{B}_i)| < 5n^3\delta, \forall i \leq n$.

Lemma 10.2. *If $A_1, \dots, A_n \in \mathrm{MALG}_\mu$ form a partition of $X, 0 \leq \alpha_i \leq 1, \sum_{i=1}^n \alpha_i = 1$ and $|\mu(A_i) - \alpha_i| < \rho, \forall i \leq n$, then there are $C_1, \dots, C_n \in \mathrm{MALG}_\mu$ which form a partition of X such that $\mu(C_i \Delta A_i) < \rho$ and $\mu(C_i) = \alpha_i, \forall i \leq n$.*

Proof. In fact, by separately considering the i for which $\mu(A_i) \geq \alpha_i$ and then those for which $\mu(A_i) < \alpha_i$, we can find C_i so that $C_i \subseteq A_i$, if

$\alpha_i \leq \mu(A_i)$, and $C_i \supseteq A_i$ if $\alpha_i > \mu(A_i)$. Indeed, by renumbering, we can assume that $A_1, \dots, A_m$ are such that $\alpha_i \leq \mu(A_i), \forall i \leq m$, and $A_{m+1}, \dots, A_n$ are such that $\mu(A_j) < \alpha_j, \forall m+1 \leq j \leq n$. Let then $C_i \subseteq A_i, i \leq m$, be such that $\mu(C_i) = \alpha_i$. Let $\lambda_i = \mu(A_i) - \alpha_i \geq 0$. Then clearly $\sum_{i=1}^m \lambda_i = \sum_{j=m+1}^n (\alpha_j - \mu(A_j))$, so we can find for each $m+1 \leq j \leq n, C_j \supseteq A_j$ with $C_j \setminus A_j \subseteq \bigcup_{j=1}^m (A_i \setminus C_i)$ and $\mu(C_j) = \alpha_j$. □

So fix a partition $C_1, \dots, C_n$ of X with $\mu(C_i) = \nu(\bar{B}_i), \forall i \leq n$, and $\mu(A_i \Delta C_i) < 5n^3\delta$. Let then $\varphi : (Y, \nu) \to (X, \mu)$ be a Borel isomorphism with $\varphi(\bar{B}_i) = C_i, \forall i \leq n$. Let φ send b to c. Then for $\gamma \in F$, $\mu(\gamma^c(C_i) \cap C_j) = \nu(\gamma^b(\bar{B}_i) \cap \bar{B}_j)$, and $\mu((\gamma^c(C_i) \cap C_j)\Delta(\gamma^c(A_i) \cap A_j)) < 10n^3\delta$, therefore $|\mu(\gamma^a(A_i) \cap A_j) - \mu(\gamma^c(A_i) \cap A_j)| \leq |\mu(\gamma^a(A_i) \cap A_j) - \nu(\gamma^b(\bar{B}_i) \cap \bar{B}_j)| + |\nu(\gamma^b(\bar{B}_i) \cap \bar{B}_j) - \mu(\gamma^c(C_i) \cap C_j)| + |\mu(\gamma^c(C_i) \cap C_j) - \mu(\gamma^c(A_i) \cap A_j)| < 5n^3\delta + 10n^3\delta < 20n^3\delta < \epsilon$, so $c \in V$. □

Thus if $a, b \in A(\Gamma, X, \mu)$,

$$a \prec b \Leftrightarrow a \text{ in the closure of the conjugacy class of } b.$$

It follows that $\prec$ on $A(\Gamma, X, \mu)$ is a special case of a familiar partial pre-ordering, present in any continuous action. If a topological group G acts continuously on a topological space P, we have the following partial pre-ordering on P:

$$x \preccurlyeq y \Leftrightarrow x \in \overline{G \cdot y} \Leftrightarrow \overline{G \cdot x} \subseteq \overline{G \cdot y}.$$

The equivalence relation induced by $\preccurlyeq$, i.e., $x \approx y \Leftrightarrow x \preccurlyeq y \;\&\; y \preccurlyeq x$, is sometimes called the *topological ergodic decomposition* of this action. It is clear from the definition that each section $\preccurlyeq^y = \{x : x \preccurlyeq y\}$ is closed and, if P is separable metrizable, then $\preccurlyeq$ is G_δ. For the particular case $G = (\mathrm{Aut}(X, \mu), w), P = (A(\Gamma, X, \mu), w)$ and the conjugacy action, we then obtain $\prec$. So we have the following fact.

Corollary 10.3. *The relation $\prec$ on $A(\Gamma, X, \mu)$ in G_δ and each section $\prec^b = \{a \in A(\Gamma, X, \mu) : a \prec b\}$ is closed in the weak topology.*

A basic property of weak containment, that will be useful later on, is stated in the next result.

Let $a_i \in A(\Gamma, X_i, \mu_i), i \in I$, where I is a countable index set, and ρ a probability Borel measure on $\prod_i X_i$ whose *marginal* on X_i is μ_i (i.e., the X_i-projection of ρ is μ_i). Denote by $\prod_i a_i$ the *product action* of Γ on $\prod_i X_i$:

$$\gamma^{\prod_i a_i}(\{x_i\}) = \{\gamma^{a_i}(x_i)\}.$$

(Note that each $A(\Gamma, X_i, \mu_i)$ is really a set of equivalence classes of Borel actions of Γ on X_i, where two actions are equivalent if they agree μ_i-a.e. Observe now that if a_i, a'_i are equivalent with respect to μ_i, for each i, then $\prod_i a_i, \prod_i a'_i$ are equivalent with respect to ρ.) If ρ is $\prod_i a_i$-invariant, then we say that ρ is a *joining* of $\{a_i\}$. In this case $\prod_i a_i \in A(\Gamma, \prod_i X_i, \rho)$. To emphasize that we are looking at $\prod_i a_i$ with respect to ρ, we also write

$(\prod_i a_i)_\rho$. A special case is when $\rho = \prod_i \mu_i$, in which case we simply write $\prod_i a_i$.

We note that the map

$$\{a_i\}_i \in \prod_i A(\Gamma, X_i, \mu_i) \mapsto \prod_i a_i \in A(\Gamma, \prod_i X_i, \prod_i \mu_i)$$

is continuous in the weak topology and, if I is finite, also continuous in the uniform topology.

Proposition 10.4. *Let $a_i \in A(\Gamma, X_i, \mu_i), i \in I$, and ρ a joining of $\{a_i\}$, so that $(\prod_i a_i)_\rho \in A(\Gamma, \prod_i X_i, \rho)$. Then*

$$a_i \prec (\prod_i a_i)_\rho, \forall i.$$

In particular,

$$a_i \prec \prod_i a_i, \forall i.$$

Proof. Fix $i \in I$. Given $A \in \mathrm{MALG}_{\mu_i}$, let $\hat{A} = \pi^{-1}(A) \in \mathrm{MALG}_\rho$, where $\pi : \prod_i X_i \to X_i$ is the projection. Clearly $\rho(\hat{A}) = \mu_i(A)$ and $\gamma^{(\prod_i a_i)_\rho}(\hat{A}) = \widehat{\gamma^{a_i}(A)}$. So given $A_1, \dots, A_n \in \mathrm{MALG}_{\mu_i}$, if $B_j = \hat{A}_j$, we have $\mu_i(\gamma^{a_i}(A_k) \cap A_\ell) = \rho(\gamma^{(\prod_i a_i)_\rho}(B_k) \cap B_\ell)$, which shows that $a_i \prec (\prod_i a_i)_\rho$. □

Note also that

$$a_i \prec b_i, \forall i \Rightarrow \prod_i a_i \prec \prod_i b_i.$$

Every $a \in A(\Gamma, X, \mu)$ gives rise to a unitary representation on $L^2(X, \mu)$, denoted by κ^a, called the *Koopman representation* associated to a. This is defined by

$$\kappa^a(\gamma)(f)(x) = f((\gamma^{-1})^a(x)).$$

The map $a \mapsto \kappa^a \in \mathrm{Rep}(\Gamma, L^2(X, \mu))$ is a homeomorphism of $(A(\Gamma, X, \mu), w)$ with the closed subspace of $\mathrm{Rep}(\Gamma, L^2(X, \mu))$ consisting of all representations $\pi \in \mathrm{Rep}(\Gamma, L^2(X, \mu))$ such that $\pi(\gamma) \in \mathrm{Aut}(X, \mu) \subseteq U(L^2(X, \mu)), \forall \gamma \in \Gamma$ (see Appendix H for a discussion of the space $\mathrm{Rep}(\Gamma, H)$ of unitary representations of a group Γ on a Hilbert space H). Let also κ_0^a be the restriction of κ^a to $L^2_0(X, \mu)$. Again $a \mapsto \kappa_0^a \in \mathrm{Rep}(\Gamma, L^2_0(X, \mu))$ is a homeomorphism of $(A(\Gamma, X, \mu), w)$ with a closed subspace of $\mathrm{Rep}(\Gamma, L^2_0(X, \mu))$. Clearly if a, b are isomorphic, κ^a (resp., κ_0^a), κ^b (resp., κ_0^b) are isomorphic. We call a, b *unitarily equivalent* (or *spectrally equivalent*) if $\kappa^a \cong \kappa^b$. This is also equivalent to $\kappa_0^a \cong \kappa_0^b$.

Note that a is ergodic (resp., weak mixing, mild mixing, mixing) iff κ_0^a is ergodic (resp., weak mixing, mild mixing, mixing) in the sense of Appendix H, **(D)**.

Recall also the notion of weak containment $\pi \prec \rho$ of unitary representations (see Appendix H). We have the following fact.

Proposition 10.5. $a \prec b \Rightarrow \kappa^a \prec \kappa^b, \kappa_0^a \prec \kappa_0^b$.

Proof. It is enough to check that $a \prec b \Rightarrow \kappa_0^a \prec \kappa_0^b$. We can assume that $a, b \in A(\Gamma, X, \mu)$ and view $A(\Gamma, X, \mu)$ as a closed subspace of $\mathrm{Rep}(\Gamma, L_0^2(X, \mu))$. If $a \prec b$, then a is in the closure of the conjugacy class of b, i.e., in $\overline{\{TbT^{-1} : T \in \mathrm{Aut}(X, \mu)\}}$, where closure is in the weak topology. Thus, since we identify a (resp., b) with κ_0^a (resp., κ_0^b) $\in \mathrm{Rep}(\Gamma, L_0^2(X, \mu))$, we have

$$\kappa_0^a \in \overline{\{U_T^0 \kappa_0^b (U_T^0)^{-1} : T \in \mathrm{Aut}(X, \mu)\}} \subseteq \overline{\{U \kappa_0^b U^{-1} : U \in U(L_0^2(X, \mu))\}}$$
$$= \overline{\{\sigma \in \mathrm{Rep}(\Gamma, L_0^2(X, \mu)) : \sigma \cong \kappa_0^b\}}.$$

Then (see the proof of H.2) $\kappa_0^a \prec \kappa_0^b$. □

Remark. In fact this argument shows that

$$a \prec b \Rightarrow \kappa^a \prec_Z \kappa^b, \kappa_0^a \prec_Z \kappa_0^b,$$

where $\prec_Z$ is the stronger notion of weak containment in the sense of Zimmer; see the Remark after Proposition H.2.

We denote by

$$i_\Gamma \in A(\Gamma, X, \mu)$$

the trivial action of Γ on $(X, \mu) : \gamma \cdot x = x$.

For the next proposition, recall the result of Jones-Schmidt [JS] that an ergodic action $a \in A(\Gamma, X, \mu)$ is not E_0-ergodic iff it admits non-trivial almost invariant sets. Here $a \in A(\Gamma, X, \mu)$ *admits non-trivial almost invariant sets* iff $\exists \delta > 0 \forall F$ finite $\subseteq \Gamma \forall \epsilon > 0 \exists A \in \mathrm{MALG}_\mu[\delta < \mu(A) < 1 - \delta$ and $\forall \gamma \in F(\mu(\gamma^a(A) \Delta A) < \epsilon)]$ iff $\exists \delta > 0 \exists \{A_n\} \subseteq \mathrm{MALG}_\mu[\delta < \mu(A_n) < 1 - \delta$, and $\forall \gamma \in F(\mu(\gamma^a(A_n) \Delta A_n) \to 0)]$. (For a survey of these notions and a proof of this result see also Hjorth-Kechris [HK3], Appendix A.) Moreover, if a is ergodic but not E_0-ergodic, we can find such almost invariant sets of any given measure $0 < c < 1$.

Proposition 10.6. *Let $a \in A(\Gamma, X, \mu)$. If $i_\Gamma \prec a$, then a admits non-trivial almost invariant sets. Moreover if a is ergodic, then $i_\Gamma \prec a \Leftrightarrow$ a is not E_0-ergodic.*

Proof. First assume $i_\Gamma \prec a$. Fix $F \subseteq \Gamma$ finite containing 1 and $\epsilon > 0$. Let also $A \in \mathrm{MALG}_\mu$ be such that $\mu(A) = 1/2$. Put $A_1 = A, A_2 = X \setminus A$. Then for any $\delta > 0$, there are $B_1, B_2 \in \mathrm{MALG}_\mu$ such that

$$\mu(B_1 \cap B_2) < \delta,$$
$$|\frac{1}{2} - \mu(B_1)|, |\frac{1}{2} - \mu(B_2)| < \delta,$$
$$\mu(B_1 \cap \gamma^a(B_2)), \mu(B_2 \cap \gamma^a(B_1)) < \delta,$$

$\forall \gamma \in F$. If we choose δ small enough, then clearly $\mu(\gamma^a(B_1) \Delta B_1) < \epsilon, \forall \gamma \in F$. So a has non-trivial almost invariant sets.

Conversely, if a is ergodic but not E_0-ergodic, then the proof of the above result of Jones-Schmidt (see again Hjorth-Kechris [HK3], A2.2) shows that for any partition $A_1, \dots, A_n$ of X, with say $\mu(A_i) = \alpha_i$, and every

$\epsilon > 0, F \subseteq \Gamma$ finite, symmetric containing 1, there is a partition $B_1, \dots, B_n$ of X with $\mu(B_i) = \alpha_i$ and $\mu(\gamma^a(B_i)\Delta B_i) < \epsilon$ for $\gamma \in F$. Thus for $i, j \leq n$,

$$|\mu(A_i \cap A_j) - \mu(\gamma^a(B_i) \cap B_j)| < \epsilon,$$

so $i_\Gamma \prec a$. □

If 1_Γ is the trivial 1-dimensional representation of Γ, then

$$1_\Gamma \prec \kappa_0^a \Leftrightarrow \kappa_0^a \text{ admits non-0 almost invariant vectors.}$$

(where a representation π *admits non-*0 *almost invariant vectors* if $\forall F \subseteq \Gamma$ finite $\forall \epsilon > 0 \exists v\ (\|\pi(\gamma)(v) - v\| < \epsilon\|v\|)$. If $i_\Gamma \prec a$, then, as $\kappa_0^{i_\Gamma} = \infty \cdot 1_\Gamma$, we have $1_\Gamma \prec \kappa_0^a$. However, $1_\Gamma \prec \kappa_0^a$ does not imply $i_\Gamma \prec a$, since there are examples of ergodic actions of $\Gamma = F_2$ with $1_\Gamma \prec \kappa_0^a$ but $i_\Gamma \not\prec a$ (see Schmidt [Sc3], 2.7 and Hjorth-Kechris [HK3], A3.2). Thus, in general, $\kappa_0^a \prec \kappa_0^b$ does not imply $a \prec b$.

(D) In Glasner-King [GK], p. 234, the authors asked for what groups Γ there is a dense conjugacy class in $(A(\Gamma, X, \mu), w)$. They note that Rudolph and Weiss pointed out that this holds for all amenable Γ and, in particular, Glasner-King asked if this is also the case for any free group. It turns out an affirmative answer for the free group case is an immediate consequence of Kechris-Rosendal [KR], 6.5, but in fact it is always true, as shown, independently, by Hjorth (unpublished) and Glasner-Thouvenot-Weiss [GTW].

Theorem 10.7 (Glasner-Thouvenot-Weiss, Hjorth). *For each countable group* Γ*, the space* $A(\Gamma, X, \mu)$ *has a weakly dense conjugacy class.*

Proof. Fix a countable dense set $\{a_n\}$ in $(A(\Gamma, X, \mu), w)$ and consider the product action $a = \prod_n a_n$ of Γ on $(X^{\mathbb{N}}, \mu^{\mathbb{N}}) : a(\gamma, \{x_n\}) = \{a_n(\gamma, x_n)\}$. Then, by 10.4, $a_n \prec a, \forall n$, so as $\{a_n\}$ is dense in $A(\Gamma, X, \mu), b \prec a$, for every $b \in A(\Gamma, X, \mu)$, i.e., an isomorphic copy of a in $A(\Gamma, X, \mu)$ has weakly dense conjugacy class. □

Also note the following result that can be proved by a similar method.

Theorem 10.8 (Glasner-King). *Let* Γ *be a countable group. The set* $\mathrm{FR}(\Gamma, X, \mu)$ *of free actions in* $A(\Gamma, X, \mu)$ *is dense* G_δ *in the weak topology.*

Proof. First note that $\mathrm{FR}(\Gamma, X, \mu)$ is G_δ. We have

$$a \in \mathrm{FR}(\Gamma, X, \mu) \Leftrightarrow \forall \gamma \neq 1 (\delta_u(\gamma^a, 1) = 1)$$

and open balls in δ_u are F_σ in the weak topology.

To check that $\mathrm{FR}(\Gamma, X, \mu)$ is dense, note that Γ admits some free action, e.g., the shift on 2^Γ if Γ is infinite or the translation action $\gamma \cdot (\delta \cdot x) = (\gamma\delta, x)$ on $\Gamma \times X$ if Γ is finite. Thus, if in the proof of 10.7 we include in the sequence $\{a_n\}$ a free action, it follows that $a = \prod_n a_n$ is free and its isomorphic copies are dense in $A(\Gamma, X, \mu)$. □

(E) We next consider factors and extensions of actions. Given two actions $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$, a *homomorphism* of a to b is a Borel map $\pi : X \to Y$ such that $\pi_*\mu = \nu$ and $\pi(\gamma^a(x)) = \gamma^b(\pi(x)),\ \mu$-a.e. $(x), \forall \gamma \in \Gamma$.

If such a homomorphism exists, then b is a *factor* of a and a an *extension* of b. We denote this by

$$b \sqsubseteq a.$$

We note that for π as above the map $\pi^*(A) = \pi^{-1}(A)$ is an embedding of MALG_ν to a Γ-invariant non-atomic σ-subalgebra of MALG_μ. Conversely, every Γ-invariant non-atomic σ-subalgebra $\mathcal{A} \subseteq \mathrm{MALG}_\mu$ gives rise to a homomorphism π from a to some b, so that π^* is an isomorphism of MALG_ν with $\mathcal{A}$ which preserves the action of Γ (see Mackey [M] or Kechris [Kec1], 3.47). Thus there is a canonical bijection between factors of a and Γ-invariant non-atomic σ-subalgebras of MALG_μ.

Simple examples of factors come from joinings. If ρ is a joining of $\{a_i\}$, clearly $a_i \sqsubseteq (\prod_i a_i)_\rho, \forall i$.

There is a canonical way to produce extensions of a given action. Let $b \in A(\Gamma, Y, \nu)$ be a given action, let (Z, σ) be a standard Borel space with a probability measure σ, not necessarily non-atomic, and let $\beta : \Gamma \times Y \to \mathrm{Aut}(Z, \sigma)$ be a Borel *cocycle* of the action b into $\mathrm{Aut}(Z, \sigma)$, i.e., a Borel map satisfying

$$\beta(\gamma_1\gamma_2, y) = \beta(\gamma_1, \gamma_2^b(y))\beta(\gamma_2, y), \text{a.e.}(y),$$

$\forall \gamma_1, \gamma_2 \in \Gamma$. Let $X = Y \times Z, \mu = \nu \times \sigma$ and define the *skew product action* $a \in A(\Gamma, X, \mu)$ of b with β, denoted by

$$b \times_\beta (Z, \sigma),$$

by

$$\gamma^a(y, z) = (\gamma^b(y), \beta(\gamma, y)(z)).$$

Clearly $b \sqsubseteq a$. Conversely a theorem of Rokhlin (see, e.g., Glasner [Gl2], 3.18) asserts that if $b \sqsubseteq a$ and a is ergodic, then $a \cong b \times_\beta (Z, \sigma)$ for some $(Z, \sigma), \beta$ as above.

We note that the argument in the proof of 10.4 clearly shows that

$$a \sqsubseteq b \Rightarrow a \prec b.$$

Moreover, it is easy to check that

$$a \sqsubseteq b \Rightarrow \kappa^a \leq \kappa^b, \kappa_0^a \leq \kappa_0^b.$$

We call a, b *weakly isomorphic*, in symbols

$$a \cong^w b$$

if $a \sqsubseteq b$ and $b \sqsubseteq a$. Thus we have

$$a \cong b \Rightarrow a \cong^w b \Rightarrow a \sim b,$$

$$a \cong^w b \Rightarrow \kappa^a \cong \kappa^b.$$

For $\Gamma = \mathbb{Z}$ and a, b free and ergodic, it is well-known that none of these implications reverses. For examples of a, b with $a \cong^w b$ but $a \not\cong b$ see, e.g., Glasner [Gl2], 7.16. Since $a \sim b$ for all such a, b, clearly $a \sim b \not\Rightarrow a \cong^w b$. Finally, weakly isomorphic actions of $\mathbb{Z}$ have equal entropies and all Bernoulli shifts on $2^{\mathbb{Z}}$ are spectrally equivalent, so the last implication does not reverse.

However, I do not know if for *every* infinite, countable group Γ, there are $a, b \in \mathrm{FRERG}(\Gamma, X, \mu)$ with $a \sim b$ but $a \not\cong b$, and also whether there are $a, b \in \mathrm{FRERG}(\Gamma, X, \mu)$ with $\kappa^a \cong \kappa^b$ but $a \not\cong^w b$.

Remark. A natural question, raised by Tsankov, is whether for a given infinite countable group Γ there is $a_\infty \in A(\Gamma, X, \mu)$ such that for any $a \in \mathrm{FRERG}(\Gamma, X, \mu), a \sqsubseteq a_\infty$. The answer is however negative. If such a_∞ existed, then $\kappa_0^a \leq \kappa_0^{a_\infty}$ for all $a \in \mathrm{FRERG}(\Gamma, X, \mu)$ and thus, by E.1 and the paragraph following it, if π is an irreducible infinite-dimensional representation of Γ, there is $a \in \mathrm{FRERG}(\Gamma, X, \mu)$ with $\pi \leq \kappa_0^a \leq \kappa_0^{a_\infty}$. If Γ is not abelian by-finite, then (see, e.g., Thoma [Th] or Hjorth [Hj3]) Γ has uncountably many non-isomorphic infinite-dimensional irreducible representations, which immediately gives a contradiction.

Consider now the case where Γ is abelian-by-finite and fix $\Delta \lhd \Gamma, [\Gamma : \Delta] < \infty, \Delta$ abelian. We use below the notation and facts of Appendix G. For any $a \in \mathrm{FRERG}(\Delta, X, \mu)$, $\mathrm{Ind}_\Delta^\Gamma(a) \in \mathrm{FRERG}(\Delta, X \times T, \mu \times \nu_T)$. Thus $\mathrm{Ind}_\Delta^\Gamma(a) \sqsubseteq a_\infty$ and so $\kappa^{\mathrm{Ind}_\Delta^\Gamma(a)} \cong \mathrm{Ind}_\Delta^\Gamma(\kappa^a) \leq \kappa^{a_\infty}$ and therefore

$$\mathrm{Ind}_\Delta^\Gamma(\kappa^a)|\Delta \leq \kappa^{a_\infty}|\Delta.$$

Let $\pi = \kappa^{a_\infty}|\Delta$. Using now Appendix F, it follows that the maximal spectral type of each $\mathrm{Ind}_\Delta^\Gamma(\kappa^a)|\Delta$ is $\ll$ the maximal spectral type of π, say ρ, and thus, by the last paragraph of Appendix G, the maximal spectral type of κ^a is $\ll \rho$. Thus we have derived that all maximal spectral types of (the Koopman representations of) free, ergodic measure preserving actions of Δ are absolutely continuous to a fixed measure ρ on $\hat{\Delta}$. But this is clearly false. For example, given any non-atomic symmetric probability measure ν on $\hat{\Delta}$ (symmetric means invariant under $x \mapsto x^{-1}$), $\varphi = \hat{\nu}$ is real positive-definite on Δ (see Appendix F) and if we consider the shift action a_ν of Δ on $(\mathbb{R}^\Delta, \mu_\varphi)$ (see Appendix D), then the maximal spectral type of a_ν is $\gg \nu$. Since, whenever ν is non-atomic, a_ν is free, ergodic (see, e.g., Cornfeld-Fomin-Sinai [CFS], 14.2.1), it follows that every non-atomic symmetric measure on $\hat{\Delta}$ is absolutely continuous to ρ, which is absurd, since $\hat{\Delta}$ is uncountable.

(F) We can also endow $A(\Gamma, X, \mu)$ with a concept of "convex combination". Let $a_1, \dots, a_n \in A(\Gamma, X, \mu)$ and $0 \leq \lambda_i \leq 1, 1 \leq i \leq n, \sum_{i=1}^n \lambda_i = 1$. Then we define, *up to isomorphism*, $b = \sum_{i=1}^n \lambda_i a_i \in A(\Gamma, X, \mu)$ as follows:

Let $\bar{X} = \bar{X} \sqcup \cdots \sqcup \bar{X}_n$, where each $\bar{X}_i$ is a copy of X, say via a copying map $x \mapsto \bar{x}_i$. Let $\bar{\mu}_i$ be the copy of μ on X_i. Define $\bar{\mu}$ on $\bar{X}$ as $\bar{\mu} = \sum_{i=1}^n \lambda_i \bar{\mu}_i$. Let $\bar{a}_i$ be the copy of a_i on X_i and let $\bar{a} = \bigcup_{i=1}^n \bar{a}_i$. Then $\sum_{i=1}^n \lambda_i a_i$ is defined to be any $b \in A(\Gamma, X, \mu)$ such that $b \cong \bar{a}$. (One can also define integrals of actions over a probability measure but we will not discuss this here.)

When each a_i is ergodic, then $\bar{X}_1, \dots, \bar{X}_n$ as above give the ergodic decomposition of $\bar{a}$. Thus the set of all convex combinations $\sum_{i=1}^n \lambda_i a_i$ for ergodic a_i gives, up to isomorphism, all the actions $a \in A(\Gamma, X, \mu)$ which have a finite ergodic decomposition. Denote the set of such actions by $\mathrm{FED}(\Gamma, X, \mu)$.

Proposition 10.9. *The set* $\mathrm{FED}(\Gamma, X, \mu)$ *of all actions in* $A(\Gamma, X, \mu)$ *that have a finite ergodic decomposition is dense in* $(A(\Gamma, X, \mu), w)$.

Proof. We can assume that $X = 2^{\mathbb{N}}$. Fix an action $a \in A(\Gamma, X, \mu)$ in order to show that a is in the weak closure of $\mathrm{FED}(\Gamma, X, \mu)$. Let $\varphi^a : X \to X^\Gamma$ be the usual orbit map $\varphi^a(x)(\gamma) = a(\gamma^{-1}, x)$. Then φ^a is a Borel embedding of the action a into the shift action s of Γ on X^Γ, which is clearly continuous. Let $\kappa = (\varphi^a)_*\mu$. Then $s \in A(\Gamma, X^\Gamma, \kappa)$ and $s \cong a$. Thus, by replacing a by s, we can assume without loss of generality that $X = 2^{\mathbb{N}}$ (as X^Γ is homeomorphic to X) and a is continuous. Consider the compact convex set of probability measures on X which are a-invariant, with the weak*-topology. The extreme points of that set are the probability measures that are a-invariant, ergodic, so by Krein-Milman the set of measures $\sum_{i=1}^m \alpha_i \mu_i$, where $0 \leq \alpha_i \leq 1, \sum_{i=1}^m \alpha_i = 1$ and each μ_i is a-invariant, ergodic, is dense in the set of a-invariant probability measures.

Since the Boolean algebra of clopen sets in $2^{\mathbb{N}}$ is dense in the measure algebra MALG_μ, a basic open nbhd of a in $A(\Gamma, X, \mu)$ can be taken to be of the form $V = \{b : \forall \gamma \in F \forall j, k \leq n |\mu(\gamma^a(A_j) \cap A_k) - \mu(\gamma^b(A_j) \cap A_k)| < \epsilon\}$, where $A_1, \dots, A_n$ is a clopen partition of $2^{\mathbb{N}}, F \subseteq \Gamma$ is finite containing 1 and $\epsilon > 0$. We need to find $b \in V \cap \mathrm{FED}(\Gamma, X, \mu)$. Find α_i, and distinct μ_i as above, such that if $\nu = \sum_{i=1}^m \alpha_i \mu_i$, then $|\mu(C) - \nu(C)| < \epsilon/3$ for any C in the Boolean algebra generated by $\{\gamma^a(A_j) : \gamma \in F, 1 \leq j \leq n\}$. Fix a Borel partition $Y_1 \cup \dots \cup Y_m = X$, so that $\mu_i(Y_i) = 1$ and each Y_i is a-invariant. Then let $\bar{Y}_i$ be a copy of Y_i with copying map $y \mapsto \bar{y}_i$, let $\bar{\mu}_i$ be the copy of μ_i on $\bar{Y}_i$, let $\bar{X} = \bar{Y}_1 \sqcup \dots \sqcup \bar{Y}_m$ be the disjoint union of the $\bar{Y}_i$, and let $\bar{\mu}$ on $\bar{X}$ be defined by $\bar{\mu} = \sum_{i=1}^m \alpha_i \bar{\mu}_i$ (so that $\bar{\mu}(\bar{Y}_i) = \alpha_i$). Let $\bar{a}_i$ be the copy of $a|Y_i$ on $\bar{Y}_i$ and let $\bar{a} = \bigcup_{i=1}^m \bar{a}_i$. Clearly $\bar{a} \in \mathrm{FED}(\Gamma, \bar{Y}, \bar{\mu})$. We will find $\bar{a} \cong b \in V$. For $A \subseteq X$, let $\bar{A} = \bigcup_{i=1}^m \overline{A \cap Y_i}$. Then $\gamma^{\bar{a}}(\bar{A}_j) = \bigcup_{i=1}^m \gamma^{\bar{a}_i}(\overline{A_j \cap Y_i}) = \bigcup_{i=1}^m \overline{\gamma^a(A_j \cap Y_i)} = \bigcup_{i=1}^m \overline{\gamma^a(A_j) \cap Y_i} = \overline{\gamma^a(A_j)}$. Clearly $\bar{A}_1, \dots, \bar{A}_m$ is a partition of $\bar{X}$ and $\bar{\mu}(\bar{A}_j) = \sum_{i=1}^m \alpha_i \bar{\mu}_i(\overline{A_j \cap Y_i}) = \sum_{i=1}^m \alpha_i \mu_i(A_j \cap Y_i) = \nu(A_j)$. Similarly,

$$\begin{aligned}
\bar{\mu}(\gamma^{\bar{a}}(\bar{A}_j) \cap \bar{A}_k) &= \bar{\mu}(\overline{\gamma^a(A_j)} \cap \bar{A}_k) \\
&= \bar{\mu}(\overline{\gamma^a(A_j) \cap A_k}) \\
&= \sum_{i=1}^m \alpha_i \bar{\mu}_i(\overline{\gamma^a(A_j) \cap A_k \cap Y_i}) \\
&= \sum_{i=1}^m \alpha_i \mu_i(\gamma^a(A_j) \cap A_k \cap Y_i) \\
&= \nu(\gamma^a(A_j) \cap A_k).
\end{aligned}$$

So $|\bar{\mu}(\bar{A}_j) - \mu(A_j)| < \epsilon/3$, thus, by Lemma 10.2, there is a Borel partition $B_1, \dots, B_m$ of X with $\mu(B_j \Delta A_j) < \epsilon/3$ and $\mu(B_j) = \bar{\mu}(\bar{A}_j)$. Fix then an isomorphism $\varphi : (\bar{X}, \bar{\mu}) \to (X, \mu)$ with $\varphi(\bar{A}_j) = B_j$ and let the image of

$\bar{a}$ under φ be $b \in A(\Gamma, X, \mu)$. Thus $\mu(\gamma^b(B_j) \cap B_k) = \bar{\mu}(\gamma^{\bar{a}}(\bar{A}_j) \cap \bar{A}_k) = \nu(\gamma^a(A_j) \cap A_k)$, so $|\mu(\gamma^a(A_j) \cap A_k) - \mu(\gamma^b(B_j) \cap B_k)| < \frac{\epsilon}{3}$. We will check that $b \in V$. We have for $\gamma \in F, j, k \leq n$,

$$\begin{aligned} |\mu(\gamma^a(A_j) \cap A_k) - \mu(\gamma^b(A_j) \cap A_k)| \leq \; & |\mu(\gamma^a(A_j) \cap A_k) - \mu(\gamma^b(B_j) \cap B_k)| \\ & + \; |\mu(\gamma^b(B_j) \cap B_k) - \mu(\gamma^b(A_j) \cap A_k)| \\ & < \; \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \end{aligned}$$

□

Note also the following simple fact:

$$\forall i \leq n (a_i \prec b_i) \Rightarrow \sum_{i=1}^{n} \lambda_i a_i \prec \sum_{i=1}^{n} \lambda_i b_i,$$

where $\sum_{i=1}^{n} \lambda_i = 1, 0 \leq \lambda_i \leq 1$.

(G) Next we discuss the relation between $A(\Gamma, X, \mu)$ and $A(\Delta, X, \mu)$ for certain pairs of groups Γ, Δ.

First, we note that the space $A(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n, X, \mu)$ with the weak (resp., uniform) topology is canonically homeomorphic to $A(\Gamma_1, X, \mu) \times \cdots \times A(\Gamma_n, X, \mu)$ with the product of the weak (resp., uniform) topologies and this homeomorphism preserves the conjugacy actions of $\mathrm{Aut}(X, \mu)$ (where in the second case we of course look at the product action).

Second, if $\Gamma \leq \Delta$, then there is a canonical continuous map

$$\pi_{\Delta,\Gamma} : A(\Delta, X, \mu) \to A(\Gamma, X, \mu)$$

defined by restriction: $\pi_{\Delta,\Gamma}(a) = a|\Delta$. Again we can use here either the weak or the uniform topology. Moreover $\pi_{\Delta,\Gamma}$ clearly preserves the conjugacy action of $\mathrm{Aut}(X, \mu)$. In case $\Delta = \Gamma * \Gamma'$, for some Γ', clearly this map is open and surjective, as in this case $A(\Delta, X, \mu)$ can be identified with the product $A(\Gamma, X, \mu) \times A(\Gamma', X, \mu)$ and $\pi_{\Delta,\Gamma}$ is simply the projection map to the first factor.

Conversely, when $\Gamma \leq \Delta$, every action of Γ gives rise to an action of Δ called the *co-induced action*. (This was brought to my attention by Adrian Ioana, who referred to Gaboriau [Ga2] in his paper Ioana [I1]. Sinelshchikov pointed out that this concept also appears in Lück [Lu], and references therein, in another, non-measure theoretic context.) We will actually consider a somewhat more general situation.

Suppose that a countable group Δ acts on a countable set T, and let $\sigma : \Delta \times T \to \mathrm{Aut}(X, \mu)$ be a cocycle of the action of Δ on T with values in $\mathrm{Aut}(X, \mu)$, i.e., a map satisfying the property

$$\sigma(\delta_1 \delta_2, t) = \sigma(\delta_1, \delta_2 \cdot t)\sigma(\delta_2, t),$$

for $\delta_1, \delta_2 \in \Delta,\ t \in T$. We define an action of Δ on (Y, ν), where $Y = X^T, \nu = \mu^T$ (the product measure), by

$$(\delta \cdot f)(t) = \sigma(\delta^{-1}, t)^{-1}(f(\delta^{-1} \cdot t)).$$

It is straightforward to check that this is indeed a measure preserving action of Δ on (Y, ν).

We note that if $\tau : \Delta \times T \to \mathrm{Aut}(X, \mu)$ is cohomologous to σ, i.e., there is $f : T \to \mathrm{Aut}(X, \mu)$ such that

$$\tau(\delta, t) = f(\delta \cdot t)\sigma(\delta, t)f(t)^{-1},$$

then the action induced by τ is isomorphic to the action induced by σ via the map $\varphi : Y \to Y$ given by $\varphi(p)(t) = f(t)(p(t))$.

If now Γ is another countable group and $a \in A(\Gamma, X, \mu)$, while $\rho : \Delta \times T \to \Gamma$ is a cocycle of the action of Δ on T with values in Γ, define the cocycle $\sigma : \Delta \times T \to \mathrm{Aut}(X, \mu)$ by $\sigma(\delta, t) = \rho(\delta, t)^a$. We call the action defined as above using this σ the *co-induced action* of a with respect to the action of Δ on T and ρ.

Consider now the case where $\Gamma \leq \Delta$ and let T be a transversal for the left cosets of Γ with $1 \in T$. Let Δ act on T by defining $\delta \cdot t$ to be the unique element of T in $\delta t\Gamma$, and let $\rho : \Delta \times T \to \Gamma$ be the cocycle defined by $\rho(\delta, t) =$ (the unique $\gamma \in \Gamma$ such that $(\delta \cdot t)\gamma = \delta t$). In that case we write

$$\mathrm{CInd}_\Gamma^\Delta(a)$$

for the co-induced action corresponding to this action and cocycle. Clearly

$$a \cong b \Rightarrow \mathrm{CInd}_\Gamma^\Delta(a) \cong \mathrm{CInd}_\Gamma^\Delta(b).$$

We note certain facts about this co-induced action (for proofs see Ioana [I1]):

(i) $a \sqsubseteq \mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$.
Indeed the map $f \mapsto f(1)$ shows that a is a factor of $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$.

(ii)

$$a \mapsto \mathrm{CInd}_\Gamma^\Delta(a)$$
$$A(\Gamma, X, \mu) \to A(\Delta, Y, \nu)$$

is a homeomorphism of $(A(\Gamma, X, \mu), w)$ onto a (necessarily G_δ) subspace of $(A(\Delta, Y, \nu), w)$. It is also a homeomorphism in the uniform topology, if $[\Delta : \Gamma] < \infty$.

(iii) Assume Γ is infinite. Then a is mixing (resp., mild mixing) iff $\mathrm{CInd}_\Gamma^\Delta(a)$ is mixing (resp., mild mixing). If $[\Delta : \Gamma] = \infty$, then $\mathrm{CInd}_\Gamma^\Delta(a)$ is weak mixing. If $[\Delta : \Gamma] < \infty$, then $\mathrm{CInd}_\Gamma^\Delta(a)$ is weak mixing iff a is weak mixing. In particular, if a is weak mixing, so is $\mathrm{CInd}_\Gamma^\Delta(a)$.

(iv) a is free iff $\mathrm{CInd}_\Gamma^\Delta(a)$ is free.

(v) If i_Γ is the trivial action of Γ on X, then $\mathrm{CInd}_\Gamma^\Delta(i_\Gamma)$ is the shift action of Δ on $X^{\Delta/\Gamma}$.

When $\Gamma \lhd \Delta$, then for any $\gamma \in \Gamma, t \in T$, $\gamma \cdot t = t$ and $\rho(\gamma, t) = t^{-1}\gamma t$, so that $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$ is given by $(\gamma \cdot f)(t) = (t^{-1}\gamma t) \cdot f(t)$. Thus if we define for each $t \in T$ the action $a_t \in A(\Gamma, X, \mu)$ by $a_t(\gamma, x) = a(t^{-1}\gamma t, x)$, then $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma = \prod_{t \in T} a_t$. Moreover, if $\Gamma \leq Z(\Delta)$, $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma = \prod_{t \in T} a = a^T$.

In particular, this shows that a being ergodic does not always imply that $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$ is ergodic. Ioana has also shown that even a being weak mixing does not always imply that $\mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$ is ergodic.

Since $a \sqsubseteq \mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$, it follows that $a \prec \mathrm{CInd}_\Gamma^\Delta(a)|\Gamma$, so in particular the set of $b \in A(\Gamma, X, \mu)$ of the form $b = c|\Gamma$, for some $c \in A(\Delta, X, \mu)$ (i.e., the set of actions of Γ that can be extended to actions of Δ) is weakly dense in $A(\Gamma, X, \mu)$. In other words, the restriction map $\pi_{\Delta,\Gamma}$ defined above has dense range in the weak topology. Moreover, if $[\Delta : \Gamma] = \infty$ the set of $b \in A(\Gamma, X, \mu)$ of the form $b = c|\Gamma$ for some $c \in \mathrm{WMIX}(\Delta, X, \mu)$ is also weakly dense in $A(\Gamma, X, \mu)$. (As we will see later, when Γ has property (T), $\mathrm{WMIX}(\Gamma, X, \mu)$ is closed nowhere dense; see 12.8.)

It follows from (ii), (iii) above and 2.7 that $\mathrm{MMIX}(\Gamma, X, \mu)$ is (co-analytic) but not Borel, if Γ has an element of infinite order. In particular, mixing, mild mixing and weak mixing are all distinct for such groups (since $\mathrm{MIX}(\Gamma, X, \mu), \mathrm{WMIX}(\Gamma, X, \mu)$ are Borel). It is unknown if this holds for *every* infinite group; see Section 11, **(C)**.

Remark. Let $\Gamma \leq \Delta$ be countable groups and denote by $A(\Gamma \uparrow \Delta, X, \mu)$ the set of actions in $A(\Gamma, X, \mu)$ that can be extended to an action of Δ. We have just seen that $A(\Gamma \uparrow \Delta, X, \mu)$ is dense in $(A(\Gamma, X, \mu), w)$. It is an interesting question to find out when the generic action of Γ extends to an action of Δ. For example, Ageev [A1] has shown that this is the case when $\Gamma = \mathbb{Z}$ and $\Delta \geq \Gamma$ is abelian. On the other hand one can see that there are cases in which the answer is negative. Let $\Gamma = F_4$ with generators $\alpha, \beta, \gamma, \delta$. The generic action a of Γ has the following properties: It is free, so in particular $\alpha^a, \beta^a, \gamma^a, \delta^a$ are distinct, and the group generated by α^a, β^a is dense in $\mathrm{Aut}(X, \mu)$ (see Section 4, **(D)**). Let φ be the automorphism of Γ such that $\varphi(\alpha) = \alpha, \varphi(\beta) = \beta, \varphi(\gamma) = \delta, \varphi(\delta) = \gamma$. Let Δ be the HNN extension of Γ corresponding to φ (i.e., the semidirect product of $\mathbb{Z}$ with Γ, where $\mathbb{Z}$ acts on Γ by $n \cdot \gamma = \varphi^n(\gamma)$). Let t be the additional generator, so that $t\gamma t^{-1} = \varphi(\gamma), \forall \gamma \in \Gamma$. If the generic action $a \in A(\Gamma, X, \mu)$ was extendible to an action b of Δ, let $\alpha^b = \alpha^a = A, \beta^b = \beta^a = B, \gamma^b = \gamma^a = C, \delta^b = \delta^a = D, t^b = T$. Then $TAT^{-1} = A, TBT^{-1} = B$. so T commutes with every element of $\langle A, B \rangle$ and thus every element of $\mathrm{Aut}(X, \mu)$, so $T = 1$. But $TCT^{-1} = D$, i.e., $C = D$, a contradiction.

We also recall from Appendix G that when $\Gamma \leq \Delta$ has finite index, we can also define for each $a \in A(\Gamma, X, \mu)$ the *induced action* $\mathrm{Ind}_\Gamma^\Delta(a)$, defined as follows: Let T, ρ be as before (for the pair Γ, Δ), ν_T be the normalized counting measure on T. Then $\mathrm{Ind}_\Gamma^\Delta(a) \in A(\Gamma, X \times T, \mu \times \nu_T)$ is defined by

$$\delta \cdot (x, t) = (\rho(\delta, t) \cdot x, \delta \cdot t).$$

If a is free (resp., ergodic), so is $\mathrm{Ind}_\Gamma^\Delta(a)$.

Finally, let Γ be a factor of Δ. The clearly there is a topological embedding (in either the weak or the uniform topology)

$$\rho_{\Gamma,\Delta} : A(\Gamma, X, \mu) \to A(\Delta, X, \mu),$$

which preserves conjugacy and such that the equivalence relation generated by a is the same as that generated by $\rho_{\Gamma,\Delta}(a)$.

(H) We conclude this section with some results and questions concerning the theory of costs. General references for this theory are Gaboriau [Ga1] and Kechris-Miller [KM]. We use the terminology and notation of [KM].

Let Γ be an infinite countable group. For each $a \in A(\Gamma, X, \mu)$, let the *cost* of a be defined by

$$C(a) = C_\mu(E_a),$$

i.e., the cost of a is the cost of the associated equivalence relation. Thus

$$a\mathrm{OE}b \Rightarrow C(a) = C(b).$$

Also $0 \leq C(a) \leq \infty$. The *cost* of Γ is defined by

$$C(\Gamma) = \inf\{C(a) : a \in \mathrm{FR}(\Gamma, X, \mu)\}.$$

Moreover, it turns out that

$$\begin{aligned} C(\Gamma) = &\ \inf\{C(a) : a \in \mathrm{FRERG}(\Gamma, X, \mu)\} \\ = &\ \min\{C(a) : a \in \mathrm{FRERG}(\Gamma, X, \mu)\}, \end{aligned}$$

see Kechris-Miller [KM], 29.1 and 29.2. We have $1 \leq C(\Gamma) \leq \infty$.

A group Γ is said to have *fixed price* if $C|\mathrm{FR}(\Gamma, X, \mu)$ is constant. It is unknown whether every infinite countable group has fixed price. However, the calculations in 18.1 of Kechris-Miller [KM] show that for each $r \in \mathbb{R}$ the set

$$A_r = \{a \in A(\Gamma, X, \mu) : C(a) < r\}$$

is analytic in $(A(\Gamma, X, \mu), w)$ and thus, in particular, C is a Baire measurable function on this space. The following is then immediate from 10.7, 10.8 and the fact that C is conjugacy invariant (see Kechris [Kec2], 8.46).

Proposition 10.10. *Let Γ be an infinite countable group. Then there is a dense G_δ set in $(\mathrm{FR}(\Gamma, X, \mu), w)$ on which the cost is constant. In particular, if C is continuous on $(\mathrm{FR}(\Gamma, X, \mu), w)$, then Γ has fixed price.*

Denote this value $C^*(\Gamma)$. Clearly $C(\Gamma) \leq C^*(\Gamma)$.

Problem 10.11. *Is $C(\Gamma) = C^*(\Gamma)$?*

Remark. Using 12.1 and 13.1 below, we also see that there is a dense G_δ set in $(\mathrm{FRERG}(\Gamma, X, \mu), w)$ on which the cost is constant. Call this fixed value $C^{**}(\Gamma)$. Clearly $C(\Gamma) \leq C^{**}(\Gamma)$. If Γ does not have property (T) then 12.2, ii) shows that $\mathrm{FRERG}(\Gamma, X, \mu)$ is dense G_δ in $(\mathrm{FR}(\Gamma, X, \mu), w)$, so $C^*(\Gamma) = C^{**}(\Gamma)$, but if Γ has property (T), 12.2, i) shows that the set $\mathrm{FRERG}(\Gamma, X, \mu)$ is closed nowhere dense in $(\mathrm{FR}(\Gamma, X, \mu), w)$ and it is not clear how the quantities $C^*(\Gamma), C^{**}(\Gamma)$ are related in this case.

In connection with 10.11, let us note the following fact.

Proposition 10.12. *Let Γ be an infinite countable group. Then*

$$\mathrm{MINCOST}(\Gamma, X, \mu) = \{a \in \mathrm{FR}(\Gamma, X, \mu) : C(a) = C(\Gamma))$$

is dense in $(\mathrm{FR}(\Gamma, X, \mu), w)$.

Proof. It has been shown by Gaboriau [Ga1] (see also Kechris-Miller [KM], 29.1) that if $b \cong \prod_n a_n$, then $C(b) \leq C(a_n), \forall n$. Let then $\{a_n\} \subseteq \mathrm{FR}(\Gamma, X, \mu)$ be dense and such that $C(\Gamma) = \inf\{C(a_n) : n \in \mathbb{N}\}$. By 10.4, if $b \cong \prod_n a_n$, then the conjugacy class of b is dense in $(\mathrm{FR}(\Gamma, X, \mu), w)$ and $C(b) = C(\Gamma)$. □

Clearly if $\mathrm{MINCOST}(\Gamma, X, \mu)$ is G_δ we have an affirmative answer to 10.11.

We next show that 10.11 has indeed a positive answer if Γ is finitely generated. The main fact is the following result which gives an estimate for the complexity of the cost function.

Theorem 10.13 (Kechris). *Let Γ be an infinite, finitely generated group. Then the cost function C is upper semicontinuous on* $(\mathrm{FR}(\Gamma, X, \mu), w)$, *i.e., for each $r \in \mathbb{R}$,*

$$A_r \cap \mathrm{FR}(\Gamma, X, \mu) = \{a \in \mathrm{FR}(\Gamma, X, \mu) : C(a) < r\}$$

is open in $(\mathrm{FR}(\Gamma, X, \mu), w)$.

Proof. We will use the arguments in 18.1 and 18.3 in Kechris-Miller [KM] together with a modification due to Melleray (see Corrections and Updates for Kechris-Miller [KM], www.math.caltech.edu/people/kechris.html).

Let $\Gamma = \langle \gamma_1, \dots, \gamma_N \rangle$ and let $\Gamma = \{g_i\}_{i \in \mathbb{N}}$ be an enumeration of Γ. Fix a countable Boolean algebra $\{A_\ell\} \subseteq \mathrm{MALG}_\mu$, which is dense in MALG_μ, let $n \mapsto ((n)_0, (n)_1)$ be a bijection of $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ and for $a \in A(\Gamma, X, \mu)$, let $\{\theta^a_n\} = \{g^a_{(n)_0} | A_{(n)_1}\}$. For $S \subseteq \mathbb{N}$, put $\Theta^a_S = \{\theta^a_n\}_{n \in S}$. Then one can see that

$$C(a) < r \Leftrightarrow \exists \text{ finite } T \subseteq \mathbb{N} \exists M [C_\mu(\Theta^a_T) + C_\mu(\{\gamma^a_i | D(\gamma^a_i, \Theta^a_{M,T})\}_{i \leq N}) < r],$$

where

$$\Theta^a_{M,T} = \{\theta_1^{\pm 1} \cdots \theta_M^{\pm 1} : \theta_i \in \Theta^a_T \vee \theta_i = \mathrm{id}, \forall i \leq M\}$$

and

$$D(\gamma^a_i, \Theta^a_{M,T}) = \{x : \gamma^a_i(x) \neq \theta(x), \forall \theta \in \Theta^a_{M,T}\}.$$

Indeed, if such T, M exist, then, as $\Theta^a_T \sqcup \{\gamma^a_i | D(\gamma^a_i, \Theta^a_{M,T})\}_{i \leq N}$ is obviously an L-graphing of E_a and has cost $< r$, clearly $C(a) < r$. Conversely, assume $C(a) < r$. Then, by the first paragraph of the proof of 18.1 in [KM], there is an L-graphing of E_a of the form $\Theta^a_S, S \subseteq \mathbb{N}$, such that $C_\mu(\Theta^a_S) = \alpha < r$. Let $\epsilon < r - \alpha$. Next notice that, since Θ^a_S is an L-graphing of E_a, for each $i \leq N$,

$$\lim_{n \to \infty} \mu(D'(\gamma^a_i, \Theta^a_{S \cap \{0, \dots, n\}}) = 0,$$

where

$$D'(\theta, \Theta) = \{x : (x, \theta(x)) \notin R_\Theta\},$$

with R_Θ the equivalence relation generated by Θ. So choose L large enough, so that if $T = S \cap \{0, \dots, L\}, \mu(D'(\gamma^a_i, \Theta^a_T)) < \frac{\epsilon}{N}, \forall i \leq N$. Furthermore, since $D'(\gamma^a_i, \Theta^a_T) = \bigcap_M D(\gamma^a_i, \Theta^a_{M,T})$, we can find M large enough so that $\mu(D(\gamma_i^a, \Theta_{M,T}^a)) < \frac{\epsilon}{N}, \forall i \leq N$. Then $C_\mu(\Theta_T^a) \leq \alpha < r - \epsilon$ and $C_\mu(\{\gamma_i^a | D(\gamma_i^a, \Theta_{M,T}^a)\}_{i \leq N}) < \frac{\epsilon}{N} \cdot N = \epsilon$, so we are done.

Now

$$C_\mu(\Theta_T^a) = \sum_{n \in T} \mu(A_{(n)_1}) = \alpha_T$$

is independent of a. So it is enough to show that for each $s \in \mathbb{R}$, finite $T \subseteq \mathbb{N}$, and $M \in \mathbb{N}$, the set

$$\{a \in \mathrm{FR}(\Gamma, X, \mu) : C_\mu(\{\gamma_i^a | D(\gamma_i^a, \Theta_{M,T}^a)\}_{i \leq N}) < s\}$$

is open in $(\mathrm{FR}(\Gamma, X, \mu), w)$.

Since

$$C_\mu(\{\gamma_i^a | D(\gamma_i^a, \Theta_{M,T}^a)\}_{i \leq N}) = \sum_{i \leq N} \mu(D(\gamma_i^a, \Theta_{M,T}^a)),$$

and $\mu : \mathrm{MALG}_\mu \to [0, 1]$ is continuous, it is enough to show that for each fixed $i \leq N$, finite $T \subseteq \mathbb{N}, M \in \mathbb{N}$,

$$\begin{gathered} a \mapsto D(\gamma_i^a, \Theta_{M,T}^a) \\ \mathrm{FR}(\Gamma, X, \mu) \to \mathrm{MALG}_\mu \end{gathered}$$

is continuous in the weak topology.

Inspecting the definition of $D(\gamma_i^a, \Theta_{M,T}^a)$, we see that there is a finite sequence $\delta_1, \ldots, \delta_k \in \Gamma$, a finite sequence $n_1, \ldots, n_\ell \in \mathbb{N}$, and a Boolean formula $\varphi(x_1, \ldots, x_k, y_{11}, \ldots, y_{1\ell}, y_{21}, \ldots, y_{2\ell}, \ldots, y_{k1}, \ldots, y_{k\ell})$, all independent of a, such that

$$\begin{aligned} D(\gamma_i^a, \Theta_{M,T}^a) = \varphi(&\mathrm{supp}(\delta_1^a), \ldots, \mathrm{supp}(\delta_k^a), \\ &\delta_1^a(A_{n_1}), \ldots, \delta_1^a(A_{n_\ell}), \\ &\delta_2^a(A_{n_1}), \ldots, \delta_2^a(A_{n_\ell}), \\ &\ldots \\ &\delta_k^a(A_{n_1}), \ldots, \delta_k^a(A_{n_\ell})). \end{aligned}$$

Recall that the Boolean operations are continuous in MALG_μ. Since $a \mapsto \delta^a(A)$ is continuous from $(A(\Gamma, X, \mu), w)$ into MALG_μ, for each $\delta \in \Gamma, A \in \mathrm{MALG}_\mu$, and since $\mathrm{supp}(\delta^a)$ is either $\emptyset$ or X on $\mathrm{FR}(\Gamma, X, \mu)$, depending on whether $\delta = 1$ or $\delta \neq 1$ (and this is the only place where we use freeness of the actions), it follows that $a \mapsto D(\gamma_i^a, \Theta_{M,T}^a)$ is continuous and the proof is complete. □

For $a, b \in \mathrm{FR}(\Gamma, X, \mu)$, put

$$a \preceq b \Leftrightarrow a \in \overline{\{c \in \mathrm{FR}(\Gamma, X, \mu) : c \mathrm{OE} b\}},$$

where closure is in the weak topology. Thus

$$a \prec b \Rightarrow a \preceq b.$$

Corollary 10.14. *Let Γ be an infinite, finitely generated group. For $a, b \in \mathrm{FR}(\Gamma, X, \mu)$,*

$$a \preceq b \Rightarrow C(a) \geq C(b).$$

In particular, if $a \in \mathrm{FR}(\Gamma, X, \mu)$ has weakly dense orbit equivalence class, restricted to free actions, e.g., if a has weakly dense conjugacy class, then $C(a) = C(\Gamma)$. Therefore,

$$C^*(\Gamma) = C(\Gamma).$$

Moreover, $\mathrm{MINCOST}(\Gamma, X, \mu)$ is dense G_δ in the weak topology and it is equal to the set of continuity points of $C|\mathrm{FR}(\Gamma, X, \mu)$.

Abért and Weiss (unpublished) have recently shown that for any infinite group Γ, the shift action s_Γ of Γ on (X, μ), where $X = 2^\Gamma$ and μ is the usual product measure, is minimum in $\prec$ among *free* actions of Γ. This implies by the above that for infinite, finitely generated groups Γ, the action s_Γ has the maximum cost among free actions of Γ.

Remark. The cost function is not upper semicontinuous in the space $(A(\Gamma, X, \mu), w)$, for any infinite Γ. Otherwise, if $a \in \mathrm{FR}(\Gamma, X, \mu)$ has weakly dense conjugacy class, then $1 \leq C(a) \leq C(b)$ for any $b \in A(\Gamma, X, \mu)$, which gives a contradiction by taking $b = i_\Gamma$ (the trivial action) for which $C(b) = 0$. On the other hand, since the map $T \mapsto \mathrm{supp}(T)$ from $\mathrm{Aut}(X, \mu)$ to MALG_μ is continuous in the uniform topology u (see Section 1), it follows that $a \mapsto \mathrm{supp}(\delta^a)$ is continuous from $(A(\Gamma, X, \mu), u)$ to $\mathrm{MALG}_\mu, \forall \delta \in \Gamma$. Then, by the argument in the proof of 10.13, the cost function C is upper semicontinuous in $(A(\Gamma, X, \mu), u)$ for any infinite, finitely generated group.

Remark. Tsankov pointed out the following fact, relating extensions of actions and cost.

If $a \in \mathrm{FR}(\Gamma, X, \mu), b \in \mathrm{FR}(\Gamma, Y, \nu)$ and $b \sqsubseteq a$, then $C(b) \geq C(a)$. In particular, $a \cong^w b \Rightarrow C(a) = C(b)$, i.e., the cost is an invariant of weak isomorphism. (Note here that Γ is not assumed to be finitely generated.)

To see this, consider the equivalence relations E_a, E_b induced by a, b, resp. Also fix a homomorphism $\pi : X \to Y$ of a to b. Given a graphing $\mathcal{G}$ of E_b (see Kechris-Miller [KM], Section 17) define a graph $\mathcal{G}' \subseteq E_a$ by

$$x\mathcal{G}'y \Leftrightarrow xE_a y \ \&\ \pi(x)\mathcal{G}\pi(y).$$

We claim that $\mathcal{G}'$ is a graphing of E_a. Indeed let $\gamma^a(x) = y$, where $\gamma \in \Gamma$. Then $\gamma^b(\pi(x)) = \pi(y)$, so there is a $\mathcal{G}$-path $\pi(x) = z_0\mathcal{G}z_1 \dots \mathcal{G}z_n = \pi(y)$. As $\mathcal{G} \subseteq E_b$, there are $\delta_0, \dots, \delta_{n-1} \in \Gamma$ such that $\delta_i^b(z_i) = z_{i+1}$. Thus $\gamma = \delta_{n-1} \cdots \delta_0$. Then $\pi(\delta_0^a(x)) = \delta_0^b(z_0) = z_1$, and so if $x_1 = \delta_0^a(x)$, we have $x\mathcal{G}'x_1$. Next $\pi(\delta_1^a(x_1)) = \delta_1^b(z_1) = z_2$, so if $x_2 = \delta_1^a(x_1), x_1\mathcal{G}'x_2$. Continuing this way, we find a $\mathcal{G}'$-path from x to y. Next note that if $d_\mathcal{G}(y)$ denotes the degree of $\mathcal{G}$ at y, then $d_{\mathcal{G}'}(x) \leq d_\mathcal{G}(\pi(x))$ (using the freeness of the action b). So $C_\mu(E_a) \leq C_\mu(\mathcal{G}') = \frac{1}{2}\int d_{\mathcal{G}'}(x)d\mu(x) \leq \frac{1}{2}\int d_\mathcal{G}(\pi(x))d\mu(x) = \frac{1}{2}\int d_\mathcal{G}(y)d\nu(y) = C_\mu(\mathcal{G})$. Since $C_\mu(E_b) = \inf_\mathcal{G} C_\mu(\mathcal{G})$ (see Kechris-Miller [KM], Section 18), it follows that $C(b) \geq C(a)$.

Note that the fact mentioned in the proof of 10.12 is a special case of this, as $b = \prod_n a_n \Rightarrow a_n \sqsubseteq b, \forall n$.

When Γ is infinite and finitely generated, 10.14 shows that C is also an invariant of weak equivalence, but I do not know if this stronger fact holds for every infinite, countable Γ.

11. Characterizations of groups with property (T) and HAP

(A) We first consider a characterization of groups with the *Haagerup Approximation Property* (HAP). We use Cherix et al. [CCJJV] as a standard reference for HAP. Moreover, other prerequisite material is developed in Appendices A–H.

The groups Γ with HAP can be defined in several equivalent ways. We will use below the following ones:

(i) Γ has HAP iff there is a unitary representation π of Γ such that $1_\Gamma \prec \pi$ and π is a c_0-representation. Recall that 1_Γ is the *trivial one-dimensional representation* and $1_\Gamma \prec \pi$ is equivalent to the existence of *non*-0 *almost invariant vectors*, i.e., the existence of a sequence $\{v_n\}$ of unit vectors such that $\|\pi(\gamma)(v_n) - v_n\| \to 0, \forall \gamma \in \Gamma$. Also π is a *mixing representation* or c_0-*representation* if the matrix coefficients $\langle \pi(\gamma)(v), v\rangle \to 0$ as $\gamma \to \infty$, for every $v \in H_\pi$.

(ii) Γ has HAP iff there is an orthogonal representation π of Γ (on a real Hilbert space) which has non-0 almost invariant vectors and is a c_0-representation (as in (i)).

It is clear that (ii) $\Rightarrow$ (i) (using complexifications as in Appendix A). For the proof of (i) $\Rightarrow$ (ii), let $\{v_n\}$ be as in i), and put $\varphi_n(\gamma) = \langle \pi(\gamma)(v_n), v_n\rangle$. Then φ_n is positive-definite, $\varphi_n(1) = 1, \varphi_n(\gamma) \to 0$ as $\gamma \to \infty$, and moreover $\mathrm{Re}\ \varphi_n(\gamma) \to 1$ as $n \to \infty$. Then $\psi_n = \mathrm{Re}\ \varphi_n$ is real positive-definite (see Berg-Christensen-Ressel [BCR], 3.1.18), $\psi_n(1) = 1, \psi_n(\gamma) \to 0$ as $\gamma \to \infty$ and $\psi_n(\gamma) \to 1$ as $n \to \infty$. Let (π_n, H_n, w_n) be the orthogonal representation given by the GNS construction for ψ_n, so that w_n is a cyclic vector and $\langle \pi_n(\gamma)(w_n), w_n\rangle = \psi_n(\gamma)$ (thus, in particular, w_n is a unit vector). Let $H = \bigoplus_n H_n, \pi = \bigoplus_n \pi_n$. Clearly $\langle \pi(\gamma)(w_n), w_n\rangle = \psi_n(\gamma) \to 1$ as $n \to \infty$, thus $\|\pi(\gamma)(w_n) - w_n\| \to 0$, and for $v \in H$, $\langle \pi(\gamma)(v), v\rangle \to 0$ as $\gamma \to \infty$, so (ii) holds.

Theorem 11.1 (Jolissaint [CCJJV]). *Let Γ be an infinite countable group. Then the following are equivalent:*

(i) Γ has the Haagerup property.

(ii) Γ has a measure preserving, mixing action which is not E_0-ergodic.

(iii) Γ has a free, measure preserving, mixing action which is not E_0-ergodic.

In particular, Γ does not have HAP *iff* $\mathrm{MIX}(\Gamma, X, \mu) \subseteq \mathrm{E_0RG}(\Gamma, X, \mu)$.

Proof. (ii) $\Rightarrow$ (iii): If $a \in A(\Gamma, X, \mu)$ satisfies (ii) and $b \in A(\Gamma, X, \mu)$ is a free, mixing action of Γ (e.g., the shift of Γ on 2^Γ), then $a \times b$ satisfies (iii).

(iii) ⇒ (i): Let $a \in A(\Gamma, X, \mu)$ satisfy (iii). Then if κ_0^a is the corresponding Koopman representation restricted to $L_0^2(X,\mu)$, κ_0^a is a c_0-representation and a is not E_0-ergodic, so $1_\Gamma \prec \kappa_0^a$ (see, e.g., 10.6 and the paragraph following it).

(i) ⇒ (ii). Since Γ has HAP, there is an orthogonal representation π of Γ on H which has non-0 almost invariant vectors $\{v_n\}$ and is a c_0-representation. By replacing π by infinitely many copies of it, we can assume that $\{v_n\}$ is orthonormal and of course H is infinite-dimensional. Consider the product space $(X,\nu) = (\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$, with μ normalized Gaussian measure on $\mathbb{R}$. Then there is an isomorphism $\theta : H \to H^{:1:}$, where $H^{:1:} = \langle p_n \rangle_{n\in\mathbb{N}}$ is the first chaos of $L_0^2(X,\nu,\mathbb{R})$ (with p_n the projection function $p_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$).

The representation π via θ gives rise to a measure preserving action a on (X,ν) such that if κ_0^a is the corresponding Koopman representation, then $H^{:1:}$ is κ_0^a-invariant and θ sends π to $\kappa_0^a|H^{:1:}$. From the Wiener chaos decomposition, using the obvious fact that if π is a c_0-representation on H, then $\pi^{\odot n}$ is a c_0-representation on $H^{\odot n}$, it follows that κ_0^a is a c_0-representation, so a is mixing. It remains to show that it is not E_0-ergodic.

Consider the Polish space $\mathrm{Rep}_O(\Gamma, H^{:1:})$ of orthogonal representations of Γ on $H^{:1:}$, viewed as a closed subspace of $O(H^{:1:})^\Gamma$, with $O(H^{:1:})$ the orthogonal group of $H^{:1:}$ (see Appendix A). Since $O(H^{:1:})$ can be identified with a closed subgroup of $\mathrm{Aut}(X,\nu)$, $\mathrm{Rep}_O(\Gamma, H^{:1:})$ is identified with a closed subspace of $\mathrm{A}(\Gamma, X, \nu)$ by identifying any $\rho \in \mathrm{Rep}_O(\Gamma, H^{:1:})$ with the unique action $b \in A(\Gamma, X, \nu)$ such that $\kappa_0^b|H^{:1:} = \rho$. Moreover the conjugacy class of ρ in $\mathrm{Rep}_O(\Gamma, H^{:1:})$ is contained, when we identify ρ and b, in the conjugacy class of b in $\mathrm{Aut}(\Gamma, X, \mu)$. Now it is easy to check that the trivial representation $\pi_0 \in \mathrm{Rep}_O(\Gamma, H^{:1:})$ is contained in the closure of the conjugacy class of (the image under θ of) π in $\mathrm{Rep}_O(\Gamma, H^{:1:})$ and, as the action corresponding to π_0 is the trivial action $i_\Gamma \in A(\Gamma, X, \nu)$, we have that $i_\Gamma \prec a$, therefore $a \notin \mathrm{E_0RG}(\Gamma, X, \mu)$ (see 10.6). □

Remark. More generally, the simple fact that is being used here is that if $\sigma, \tau \in \mathrm{Rep}_O(\Gamma, H^{:1:})$ and $c, d \in A(\Gamma, X, \nu)$ are the actions such that $\sigma = \kappa_0^c|H^{:1:}$ and $\tau = \kappa_0^d|H^{:1:}$, then

$$\sigma \prec_Z \tau \Rightarrow c \prec d,$$

where $\sigma \prec_Z \tau$ is the obvious orthogonal analog of the notion of weak containment in the sense of Zimmer (see the Remark after H.2), which is equivalent to the property that σ is in the closure of the conjugacy class of τ in $\mathrm{Rep}_O(\Gamma, H^{:1:})$.

There is also another way to see that a, in the proof of 11.1 (i) ⇒ (ii), is not E_0-ergodic (following the method of Connes-Weiss [CW]). What one needs to show is that if π is an orthogonal representation of Γ on $H^{:1:}$, a is the action in $A(\Gamma, X, \nu)$ with $\pi = \kappa_0^a|H^{:1:}$, and if π has non-0 almost invariant vectors $\{q_n\}$, then a has non-trivial almost invariant sets.

To prove this, put $A_n = \{x \in X : q_n(x) \geq 0\}$. Since every function in $H^{:1:}$ is a centered Gaussian random variable, clearly $\nu(A_n) = \frac{1}{2}$. We will verify that $\nu(\gamma^a(A_n)\Delta A_n) \to 0, \forall \gamma \in \Gamma$, or equivalently $\nu(\gamma^a(A_n) \setminus A_n) \to 0, \forall \gamma \in \Gamma$. Fix $\gamma \in \Gamma$. Since $\kappa_0^a(\gamma)(H^{:1:}) = H^{:1:}$, we have $\kappa_0^a(\gamma)(q_n) = (\cos a_n)q_n + (\sin a_n)r_n, 0 \leq a_n < 2\pi$, where $r_n \in H^{:1:}$ is a unit vector perpendicular to the unit vector q_n. Then q_n, r_n are independent centered normalized Gaussian random variables. Clearly $\cos a_n = \langle \pi(\gamma)(q_n), q_n \rangle \to 1$ as $n \to \infty$.

Now

$$x \in \gamma \cdot A_n \setminus A_n \Leftrightarrow [\kappa_0^a(\gamma)(q_n)(x) \geq 0 \text{ and } q_n(x) < 0]$$
$$\Leftrightarrow [(\cos a_n)q_n(x) + (\sin a_n)r_n(x) \geq 0 \text{ and } q_n(x) < 0],$$

thus, since q_n, r_n are independent, normalized Gaussian variables,

$$\nu(\gamma \cdot A_n \setminus A_n) = \mu^2(\{(q,r) \in \mathbb{R}^2 : (\cos a_n)q + (\sin a_n)r \geq 0 \text{ and } q < 0\}),$$

so $\nu(\gamma \cdot A_n \setminus A_n) \to 0$.

(B) We finally prove the following characterization of groups with property (T), which preceded historically the characterization 11.1 of groups with HAP. Recall that Γ has *property* (T) if for every unitary representation π of Γ, $1_\Gamma \prec \pi$ implies $1_\Gamma \leq \pi$.

Theorem 11.2 (Connes-Weiss [CW], Schmidt [Sc3], Glasner-Weiss [GW]). *Let Γ be an infinite countable group. Then the following are equivalent:*

(i) Γ does not have property (T).

(ii) There is a measure preserving, ergodic action of Γ which is not E_0-ergodic.

(iii) There is a free, measure preserving, ergodic action of Γ which is not E_0-ergodic.

(iv) There is a measure preserving, weak mixing action of Γ which is not E_0-ergodic.

(v) There is a free, measure preserving, weak mixing action of Γ which is not E_0-ergodic.

In particular, Γ has property (T) *iff* $\mathrm{WMIX}(\Gamma, X, \mu) \subseteq \mathrm{E_0RG}(\Gamma, X, \mu)$ *iff* $\mathrm{ERG}(\Gamma, X, \mu) = \mathrm{E_0RG}(\Gamma, X, \mu)$.

Proof. (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (i). Suppose a is a measure preserving, ergodic action of Γ on (X, μ) which is not E_0-ergodic. Then if κ_0^a is the corresponding Koopman representation on $L_0^2(X, \mu)$, $1_\Gamma \prec \pi_0^a$ (since the action is not E_0-ergodic) and $1_\Gamma \not\leq \kappa_0^a$, since the action is ergodic. So Γ does not have property (T).

(i) $\Rightarrow$ (iv) $\Rightarrow$ (v). As in the proof of 11.1, it is enough, assuming Γ does not have property (T), to find an action which is measure preserving, weak mixing and not E_0-ergodic.

By the definition of property (T), there is a unitary representation ρ of Γ on H_ρ such that $1_\Gamma \prec \rho$ but $1_\Gamma \not\leq \rho$. Let $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ and, since $1_\Gamma \prec \rho$, fix a sequence of unit vectors $v_1, v_2, \dots$ such that $\|\rho(\gamma)(v_n) - v_n\|^2 < \frac{1}{2^n}$ for $\gamma \in \{\gamma_1, \dots, \gamma_n\}$. Let

$$\varphi(\gamma) = \sum_n (n\|\rho(\gamma)(v_n) - v_n\|^2) < \infty.$$

Since $\|\rho(\gamma)(v_n) - v_n\|^2 = 2 - 2 \text{ Re } \langle \rho(\gamma)(v_n), v_n\rangle$, clearly $\varphi_n(\gamma) = \|\rho(\gamma)(v_n) - v_n\|^2$ is negative-definite and so is φ (also note that $\varphi(1) = 0$). Recall that a real function θ on Γ is *negative-definite* if $\theta(\gamma) = \theta(\gamma^{-1})$ and moreover we have that $\sum_{1 \leq i,j \leq m} \alpha_i \alpha_j \theta(\gamma_i^{-1}\gamma_j) \leq 0$, for all $\gamma_1, \dots, \gamma_m \in \Gamma$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ with $\sum_{i=1}^m \alpha_i = 0$.

We next show that φ is unbounded. First notice that for each n, there is δ_n such that $\|\rho(\delta_n)(v_n) - v_n\|^2 > \frac{1}{4}$. Indeed, otherwise $\|\rho(\gamma)(v_n) - v_n\| \leq \frac{1}{2}, \forall \gamma \in \Gamma$, so the closure of the convex hall of $\{\rho(\gamma)(v_n) : \gamma \in \Gamma\}$ is contained in the ball of radius $\frac{1}{2}$ around v_n. Then if $\bar{v}_n$ is the unique element of least norm in this closed convex set, clearly $\bar{v}_n \neq 0$ and $\bar{v}_n$ is Γ-invariant, contradicting that $1_\Gamma \not\lesssim \rho$. It follows that $\varphi(\delta_n) > \frac{n}{4}$, so φ is unbounded.

Consider now the real positive-definite function $\psi_n = e^{-\varphi/n}$, and let (π_n, H_n, w_n) be the orthogonal representation of Γ given by the GNS construction for ψ_n, so that w_n is a cyclic vector and $\langle \pi_n(\gamma)(w_n), w_n\rangle = \psi_n(\gamma)$. (In particular, as $\psi_n(1) = 1, w_n$ is a unit vector.) Let $H = \bigoplus_n H_n, \pi = \bigoplus_n \pi_n$. As $\psi_n(\gamma) \to 0$, for $n \to \infty$, the sequence $\{w_n\}$ witnesses that H has non-0 almost invariant vectors.

Next as in the proof of (i) $\Rightarrow$ (ii) for 11.1, whose notation we use below, let $a \in A(\Gamma, X, \nu)$ be the corresponding action. As in that proof, a is not E_0-ergodic, so it remains to show that it is weak mixing, i.e., that the corresponding Koopman representation κ_0^a is weak mixing. We first recall the following equivalent characterizations for a unitary representation σ of Γ on H_σ to be weak mixing (see Bergelson-Rosenblatt [BR], 1.9 and 4.1, Glasner [Gl2], 3.2, 3.5 and 1.52):

(a) σ has no non-0 finite-dimensional subrepresentations,

(b) $\forall \xi_1, \dots, \xi_n \in H_\sigma \forall \epsilon > 0 \exists \gamma \in \Gamma \forall i, j (|\langle \sigma(\gamma)(\xi_i), \xi_j\rangle| < \epsilon)$,

(c) If $\mathcal{M}$ is the mean on the weakly almost periodic functions on Γ (see Glasner [Gl2], 1.51), then for any $v \in H_\sigma$, if $\psi(\gamma) = \langle \sigma(\gamma)(v), v\rangle$, then $\mathcal{M}(|\psi|^2) = 0$. Recall here that every positive-definite function is weakly almost periodic and $\psi, |\psi|^2$ as above are positive-definite. Moreover, for any positive-definite function ω,

$$\mathcal{M}(\omega) = \inf\{\sum \alpha_i \alpha_j \omega(\delta_i^{-1}\delta_j) : \alpha_i \geq 0, \sum_{i=1}^m \alpha_i = 1, \delta_1, \dots, \delta_m \in \Gamma\}$$

(see Godement [Go], THEOREME 14).

Using (b) for $\sigma = \kappa_0^a$ and the Wiener chaos decomposition, it is easy to see that it is enough to show that $\kappa_0^a | H_{\mathbb{C}}^{:1:}$ is weak mixing, which (by either (a) or (c)) is reduced to showing that each $\pi_n + i \cdot \pi_n$ is weak mixing. For this it is clearly enough to show that $\mathcal{M}(\psi_n^2) = \mathcal{M}(e^{-2\varphi/n}) = 0$. This is a general fact about unbounded negative-definite functions (see Jolissaint [Jo1], 4.4 or [Jo3], 2.1) but it is easy to adapt the proof in our case without using any general theory.

We will show that $\mathcal{M}(e^{-\varphi}) = 0$, the proof for $e^{-2\varphi/n}$ being identical. First note that we can represent φ in the form $\varphi(\gamma^{-1}\delta) = \|f(\gamma) - f(\delta)\|^2$ for some function $f : \Gamma \to F, F$ a Hilbert space, with $f(1) = 0$. Indeed, let $F = \bigoplus_n H_\rho$ be the direct sum of infinitely many copies of H_ρ and put $f(\gamma) = \bigoplus_n \sqrt{n}(\rho(\gamma)(v_n) - v_n)$. In particular, $\varphi(\gamma) = \|f(\gamma)\|^2$, therefore

$$\begin{aligned}\varphi(\gamma^{-1}\delta) &= \|f(\gamma) - f(\delta)\|^2 \\ &\geq (\|f(\gamma)\| - \|f(\delta)\|)^2 \\ &= |\sqrt{\varphi(\gamma)} - \sqrt{\varphi(\delta)}|^2.\end{aligned}$$

Recalling that

$$\mathcal{M}(e^{-\varphi}) = \inf\{\sum \alpha_i\alpha_j e^{-\varphi(\delta_i^{-1}\delta_j)} : \alpha_i \geq 0, \sum_{i=1}^m \alpha_i = 1,\ \delta_1, \cdots, \delta_m \in \Gamma\},$$

fix $\epsilon > 0$ and find m large enough so that if $a_i = \frac{1}{m}$, then $\sum_{i=1}^m \alpha_i^2 = \frac{1}{m} < \epsilon/2$. Then inductively define $\delta_1 = 1, \ldots, \delta_m$, so that, if $1 \leq j < i \leq m$, $\sqrt{\varphi(\delta_i)} > \sqrt{\log(\frac{2}{\epsilon})} + \sqrt{\varphi(\delta_j)}$, and thus $\varphi(\delta_i^{-1}\delta_j) > \log(2/\epsilon)$, $e^{-\varphi(\delta_i^{-1}\delta_j)} < \epsilon/2$, and therefore $\sum \alpha_i\alpha_j e^{-\varphi(\delta_i^{-1}\delta_j)} < \epsilon$. □

Remark. In the proof of (i) ⇒ (iv) in 11.2, we have started with a representation ρ of a countable group Γ such that $1_\Gamma \prec \rho$ and $1_\Gamma \not\leq \rho$ (i.e., ρ is ergodic). We then constructed a real negative-definite φ which is unbounded and then considered the real positive-definite $\psi_n = e^{-\varphi/n}$ and the corresponding orthogonal representation π_n and finally the direct sum $\pi = \bigoplus_n \pi_n$. If $\tau = \pi \oplus i \cdot \pi$ is the complexification of π, then $1_\Gamma \prec \tau$ and the unboundedness of φ was used to show that $\pi_n \oplus i \cdot \pi_n$ is weak mixing, thus so is τ. So by starting with an ergodic ρ with $1_\Gamma \prec \rho$ one ends up with a weak mixing τ with $1_\Gamma \prec \tau$.

(C) Schmidt [Sc4] calls groups Γ for which

$$\mathrm{WMIX}(\Gamma, X, \mu) = \mathrm{ERG}(\Gamma, X, \mu)$$

groups with *property* (W). He characterizes infinite such groups as those that admit no non-trivial finite-dimensional unitary representations (such groups are called *minimally almost periodic*). Dye [Dy] has shown that there are amenable groups with property (W). Schmidt [Sc4] also considers groups with *property* (M) and *property* (S), characterized by the equalities $\mathrm{MMIX}(\Gamma, X, \mu) = \mathrm{ERG}(\Gamma, X, \mu)$, and $\mathrm{MIX}(\Gamma, X, \mu) = \mathrm{ERG}(\Gamma, X, \mu)$, resp. It is apparently unknown whether (countable infinite) such groups exist.

12. The structure of the set of ergodic actions

(A) First we note the following fact.

Proposition 12.1. *Let* Γ *be a countable group. The sets* $\mathrm{ERG}(\Gamma, X, \mu)$, $\mathrm{WMIX}(\Gamma, X, \mu)$ *are* G_δ *in the weak topology of* $A(\Gamma, X, \mu)$.

Proof. First consider $\mathrm{ERG}(\Gamma, X, \mu)$. One way to prove this is to use Section 10, **(B)**. Take $(X, \mu) = (\mathbb{T}, \lambda)$. The ergodic (relative to the shift s) measures in $\mathrm{SIM}(\Gamma)$ are the extreme points of $\mathrm{SIM}(\Gamma)$, thus the set $\mathrm{ESIM}(\Gamma)$ of ergodic shift-invariant measures is G_δ and so is $\mathrm{ESIM}_\lambda(\Gamma) = \mathrm{SIM}_\lambda(\Gamma) \cap \mathrm{ESIM}(\Gamma)$. Since Φ_λ is continuous and $\Phi_\lambda^{-1}(\mathrm{ESIM}_\lambda(\Gamma)) = \mathrm{ERG}(\Gamma, \mathbb{T}, \lambda)$ the proof is complete.

An alternative way to prove this is as follows: The action $a \in A(\Gamma, X, \mu)$ is ergodic iff the corresponding Koopman representation κ_0^a on $L_0^2(X, \mu)$ is ergodic, i.e., for every $v \in L_0^2(X, \mu)$, if $\varphi_v(\gamma) = \langle \kappa_0^a(\gamma)(v), v \rangle$, then $\mathcal{M}(\varphi_v) = 0$, where $\mathcal{M}$ is the mean on the weakly almost periodic functions on Γ (see Glasner [Gl2], 3.2, 3.9, 3.10). Now recall that

$$\mathcal{M}(\varphi_v) = \inf\{\sum \alpha_i \alpha_j \varphi_v(\gamma_i^{-1}\gamma_j) : \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1, \gamma_1, \dots, \gamma_n \in \Gamma\}.$$

Moreover, it is clear that we can restrict v here to any countable dense set $D \subseteq L_0^2(X, \mu)$. Thus, restricting below ϵ to rationals,

$$\begin{aligned} a \in \mathrm{ERG}(\Gamma, X, \mu) \Leftrightarrow \quad & \forall v \in D \forall \epsilon > 0 \exists (\alpha_i)_{i=1}^n, (\gamma_i)_{i=1}^n \\ & [\alpha_i \geq 0, \sum \alpha_i = 1, \sum \alpha_i \alpha_j \varphi_v(\gamma_i^{-1}\gamma_j) < \epsilon], \end{aligned}$$

so clearly $\mathrm{ERG}(\Gamma, X, \mu)$ is G_δ.

A similar argument works for WMIX. By Bergelson-Rosenblatt [BR], 4.1, we have

$$a \in \mathrm{WMIX}(\Gamma, X, \mu) \Leftrightarrow \forall v_1, \dots, v_n \in D \forall \epsilon \exists \gamma (|\langle \kappa_0^a(\gamma)(v_i), v_i \rangle| < \epsilon, \forall i \leq n).$$

Alternatively we can use the fact that

$$a \in \mathrm{WMIX}(\Gamma, X, \mu) \Leftrightarrow a \times a \in \mathrm{ERG}(\Gamma, X \times X, \mu \times \mu)$$

and $a \mapsto a \times a$ is continuous in the weak topology. □

Note also that $\mathrm{ERG}(\Gamma, X, \mu)$ has no interior in w. To see this take a non-ergodic action b and any $a \in A(\Gamma, X, \mu)$. We have seen in 10.4 that a is in the closure of the set of isomorphic copies of $a \times b$, which is clearly non-ergodic. Thus $\mathrm{ERG}(\Gamma, X, \mu)$ has no interior.

(B) We have the following important dichotomy, originally formulated in Glasner-Weiss [GW] in terms of the structure of the set of ergodic measures in $\mathrm{SIM}(\Gamma)$.

Theorem 12.2 (Glasner-Weiss [GW]). *i) If a countable group* Γ *has property* (T), *then the set of ergodic actions* $\mathrm{ERG}(\Gamma, X, \mu)$ *is closed in the weak topology of* $A(\Gamma, X, \mu)$ *(and has no interior).*

ii) If a countable group Γ *does not have property* (T), *then* $\mathrm{ERG}(\Gamma, X, \mu)$ *is dense* G_δ *in the weak topology of* $A(\Gamma, X, \mu)$.

Proof. i) Given $a \in A(\Gamma, X, \mu)$ consider the corresponding Koopman unitary representation κ_0^a on $L_0^2(X, \mu)$. Then $a \in \mathrm{ERG}(\Gamma, X, \mu) \Leftrightarrow \kappa_0^a$ has no invariant non-0 vectors. Recall that a *Kazhdan pair* for Γ is a pair (Q, ϵ), where $Q \subseteq \Gamma, Q$ finite, $\epsilon > 0$, such that in any unitary representation π of Γ, if there is a vector v with $\|\pi(\gamma)(v) - v\| < \epsilon\|v\|, \forall \gamma \in Q$, then there is a non-0 invariant vector. It is well-known (see Bekka-de la Harpe-Valette [BdlHV]) that Γ has property (T) iff it admits a Kazhdan pair. Then for a Kazhdan pair (Q, ϵ),

$$\begin{aligned} a \notin \mathrm{ERG}(\Gamma, X, \mu) \ \Leftrightarrow \ & \exists v \in L_0^2(X, \mu) \forall \gamma \in Q \\ & \|\kappa_0^a(\gamma)(v) - v\| < \epsilon\|v\|, \end{aligned}$$

which clearly shows that $A(\Gamma, X, \mu) \setminus \mathrm{ERG}(\Gamma, X, \mu)$ is open in w.

ii) Using 10.9, it is enough to show that $\mathrm{ERG}(\Gamma, X, \mu)$ is dense in the set $\mathrm{FED}(\Gamma, X, \mu)$. Again for that it is enough to show that if $a_1, a_2 \in \mathrm{ERG}(\Gamma, X, \mu), \lambda_1 + \lambda_2 = 1, 0 \le \lambda_i \le 1$, then $\lambda_1 a_1 + \lambda_2 a_2 \prec b$ for some ergodic b. Because then by induction on n it is easy to see (using the paragraph after 10.9) that if $a_i \in \mathrm{ERG}(\Gamma, X, \mu), 0 \le \lambda_i \le 1$, then $\sum_{i=1}^n \lambda_i a_i \prec b$ for some ergodic b.

First notice that there is an ergodic a such that $a_1, a_2 \prec a$. This follows from 10.4, by taking ρ to be an ergodic joining of a_1, a_2 (see also the proof of 13.1). So $\lambda_1 a_1 + \lambda_2 a_2 \prec \lambda_1 a + \lambda_2 a$, thus it is enough to show for ergodic a and $0 \le \lambda_1, \lambda_2 \le 1, \lambda_1 + \lambda_2 = 1$, that $\lambda_1 a + \lambda_2 a \prec b$ for some ergodic b. Since Γ does not have property (T), by 11.2 there is $a_0 \in \mathrm{WMIX}(\Gamma, X, \mu) \setminus \mathrm{E_0RG}(\Gamma, X, \mu)$. We will in fact show that

$$\lambda_1 a + \lambda_2 a \prec a_0 \times a.$$

Since a_0 is not E_0-ergodic, find a sequence A_n^0 of Borel sets such that $\mu(A_n^0) = \lambda_1$ (so $\mu(X \setminus A_n^0) = \lambda_2$) and $\mu(\gamma^{a_0}(A_n^0) \Delta A_n^0) \to 0, \forall \gamma \in \Gamma$ (i.e., $\{A_n^0\}$ is almost invariant for a_0; see Jones-Schmidt [JS] or Hjorth-Kechris [HK3], A2.2, A2.3).

Let $Y = X_1 \sqcup X_2$ be the disjoint union of two copies of X and let $\nu = \lambda_1 \mu_1 + \lambda_2 \mu_2$, where μ_i is the copy of μ on X_i. Thus $\nu(X_i) = \lambda_i$. Let c be the union of the copy of a on X_1 and the copy of a on X_2. Thus $c \cong \lambda_1 a + \lambda_2 a$. We will check that $c \prec a_0 \times a$. Indeed given Borel sets $A_1, \dots, A_m$ in Y, let $A_i^{(j)} = A_i \cap X_j \subseteq X$. Put

$$B_i^{(n)} = (A_n^0 \times A_i^{(1)}) \cup ((X \setminus A_n^0) \times A_i^{(2)}).$$

Fix $F \subseteq \Gamma$ finite and $\epsilon > 0$. Then we claim that if n is large enough

$$|\nu(\gamma^c(A_i) \cap A_k) - (\mu \times \mu)(\gamma^{a_0 \times a}(B_i^{(n)}) \cap B_k^{(n)})| < \epsilon.$$

Indeed

$$\begin{aligned} \nu(\gamma^c(A_i) \cap A_k) \ = \ & \lambda_1 \mu(\gamma^a(A_i^{(1)}) \cap A_k^{(1)}) + \\ & \lambda_2 \mu(\gamma^a(A_i^{(2)}) \cap A_k^{(2)}) \end{aligned}$$

and if n is large enough

$$\gamma^{a_0\times a}(B_i^{(n)})$$

is very close to $(A_n^0\times\gamma^a(A_i^{(1)}))\cup((X\setminus A_n^0)\times\gamma^a(A_i^{(2)}))$ in the measure algebra of $\mu\times\mu$, thus $\gamma^{a_0\times a}(B_i^{(n)})\cap B_k^{(n)}$ is very close to $(A_n^0\times(\gamma^a(A_i^{(1)})\cap A_k^{(1)}))\cup((X\setminus A_n^0)\times(\gamma^a(A_i^{(2)})\cap A_k^{(2)}))$, which has $\mu\times\mu$-measure equal to

$$\lambda_1\mu(\gamma^a(A_i^{(1)})\cap A_k^{(1)})+\lambda_2\mu(\gamma^a(A_i^{(2)})\cap A_k^{(2)}),$$

so we are done. □

Remark. A similar argument as in the proof of 12.2, i) above can be also given to show the result of Glasner-Weiss that if Γ has property (T), then the set ESIM(Γ) of ergodic measures in SIM(Γ) (see Section 10, **(B)**) is closed in SIM(Γ). One only needs to notice that in this proof v can be restricted to any dense set in $L^2_0(X,\mu)$ and that if X is compact, the continuous functions in $L^2_0(X,\mu)$ are dense in $L^2_0(X,\mu)$.

Remark. Note that 12.2, ii) is equivalent to the fact that ESIM(Γ) is dense G_δ in SIM(Γ), which is what is proved in Glasner-Weiss [GW]. Indeed, using the Glasner-King [GK] correspondence described in Section 10, **(B)**, we have that $(h,a)\in\Phi^{-1}(\mathrm{ESIM}(\Gamma))\Leftrightarrow a\in\mathrm{ERG}(\Gamma,\mathbb{T},\lambda)$ and so ESIM(Γ) is dense G_δ in SIM(Γ) iff $\Phi^{-1}(\mathrm{ESIM}(\Gamma))$ is dense G_δ in $H\times A(\Gamma,\mathbb{T},\lambda)$ iff $\mathrm{ERG}(\Gamma,\mathbb{T},\lambda)$ is dense G_δ in $A(\Gamma,\mathbb{T},\lambda)$.

(C) On the other hand, for *any* group Γ, there are very few E_0-ergodic actions.

Proposition 12.3. *For any countable group* Γ*, the set* $\mathrm{E_0RG}(\Gamma,X,\mu)$ *is meager in* $(A(\Gamma,X,\mu),w)$*.*

Proof. The set of actions $I(\Gamma,X,\mu)$ which admit almost invariant sets of measure 1/2, i.e., $\forall F$ finite $\subseteq\Gamma\forall\epsilon>0\exists A(\mu(A)=\frac{1}{2}\ \&\ \forall\gamma\in F\mu(\gamma\cdot A\Delta A)<\epsilon)$ is a G_δ set. Let $\{a_n\}_{n=1}^\infty$ be dense in $A(\Gamma,X,\mu)$ and let $a_0\in A(\Gamma,X,\mu)$ have an invariant set of measure $\frac{1}{2}$. If $a_\infty=\prod_{n=0}^\infty a_n$, then the isomorphic copies of a_∞ are dense in $(A(\Gamma,X,\mu),w)$, by 10.4, and clearly they belong to $I(\Gamma,X,\mu)$, so $I(\Gamma,X,\mu)$ is dense G_δ. Clearly $I(\Gamma,X,\mu)\cap\mathrm{E_0RG}(\Gamma,X,\mu)=\emptyset$ (see the paragraph preceding 10.6), so $\mathrm{E_0RG}(\Gamma,X,\mu)$ is meager. □

Thus we also have another dichotomy.

Corollary 12.4. *i) If a countable group* Γ *has property* (T)*, the set of ergodic but not* E_0*-ergodic actions is empty.*

ii) If a countable group Γ *does not have property* (T)*, the set of ergodic but not* E_0*-ergodic actions is dense* G_δ *in* $(A(\Gamma,X,\mu),w)$*.*

Proof. i) follows from 11.2. For ii) notice that

$$\mathrm{ERG}(\Gamma,X,\mu)\setminus\mathrm{E_0RG}(\Gamma,X,\mu)=\mathrm{ERG}(\Gamma,X,\mu)\cap I(\Gamma,X,\mu)$$

(as in the proof of 12.3) and use 12.2. □

We finally note the following corollary of 12.2.

Corollary 12.5. *Let* Γ *be a countable group. Then the following are equivalent:*

(i) Γ *has property* (T),

(ii) $\mathrm{ERG}(\Gamma, X, \mu)$ *is closed in* $(A(\Gamma, X, \mu), w)$,

(iii) $\mathrm{ERG}(\Gamma, X, \mu)$ *is* F_σ *in* $(A(\Gamma, X, \mu), w)$.

Proof. (i) $\Rightarrow$ (ii) is 12.2, i) and (ii) $\Rightarrow$ (iii) is obvious. Assume now (iii). Then as the complement of $\mathrm{ERG}(\Gamma, X, \mu)$ is dense G_δ in $A(\Gamma, X, \mu)$, 12.2 ii) shows that Γ has property (T), i.e., (i) holds. □

This corollary provides an abstract "descriptive" way of thinking about property (T) for a countable group Γ, in terms of the structure of the set of its ergodic actions. It has the following immediate consequence: Suppose $U_\Gamma \subseteq A(\Gamma, X, \mu)$ is a G_δ property (set) of measure preserving actions of Γ (in the weak topology), which is satisfied by all non-ergodic actions, i.e., $A(\Gamma, X, \mu) \setminus \mathrm{ERG}(\Gamma, X, \mu) \subseteq U_\Gamma$. Then Γ has property (T) if U_Γ implies (thus is equivalent to) non-ergodicity.

For example, fix finite $Q \subseteq \Gamma, \epsilon > 0$, and call a set $A \in \mathrm{MALG}_\mu$, (Q, ϵ)-*invariant* for an action a, if

$$\forall \gamma \in Q(\mu(\gamma^a(A)\Delta A) < \epsilon\mu(A)(1 - \mu(A)).$$

Recall that (Q, ϵ) is a Kazhdan pair for Γ if for every unitary representation $\pi : \Gamma \to U(H)$, if there is $v \in H$ which is (Q, ϵ)-*invariant*, i.e., $\|\pi(\gamma)(v) - v\| < \epsilon\|v\|, \forall \gamma \in Q$, then there exists $v \neq 0$ which is invariant, i.e, $\pi(\gamma)(v) = v, \forall \gamma \in \Gamma$. If (Q, ϵ) is a Kazhdan pair for Γ and $a \in A(\Gamma, X, \mu)$ admits a non-trivial (Q, ϵ^2)-invariant set A, then in the Koopman representation κ_0^a we have, for $v = \chi_A - \mu(A)$,

$$\|\kappa_0^a(\gamma)(v) - v\|^2 = \mu(\gamma^a(A)\Delta A),$$
$$\|v\|^2 = \mu(A)(1 - \mu(A)),$$

thus $\|\kappa_0^a(\gamma)(v) - v\| < \epsilon\|v\|, \forall \gamma \in Q$, and so κ_0^a admits a non-0 invariant vector, therefore a is not ergodic. So this leads to the following conclusion.

Corollary 12.6. *Let* Γ *be a countable group. Then the following are equivalent:*

(i) Γ *has property* (T),

(ii) There is finite $Q \subseteq \Gamma, \epsilon > 0$, *such that every* $a \in A(\Gamma, X, \mu)$ *which admits a* (Q, ϵ)*-invariant set has a non-trivial invariant set, i.e., is not ergodic.*

Proof. Let

$$U_{\Gamma,Q,\epsilon} = \{a \in A(\Gamma, X, \mu) : a \text{ admits a } (Q, \epsilon)\text{-invariant set}\}.$$

Then $U_{\Gamma,Q,\epsilon}$ is open in $(A(\Gamma, X, \mu), w)$ and $A(\Gamma, X, \mu) \setminus \mathrm{ERG}(\Gamma, X, \mu) \subseteq U_{\Gamma,Q,\epsilon}$. So if (ii) holds for some $(Q, \epsilon), \Gamma$ has property (T).

Conversely, if Γ has property (T) with Kazhdan pair (Q, ϵ), by the paragraph preceding 12.6, (Q, ϵ^2) satisfies (ii). □

(D) In Glasner [Gl2], 13.20 a characterization of groups failing to have property HAP was obtained, analogous to that for property (T) groups that was established in Glasner-Weiss [GW]. This characterization asserted that the closure of the mixing shift-invariant measures on $\mathbb{T}^\Gamma$ is contained in the ergodic shift-invariant measures. By a combination of the arguments used in the proof of 12.2 and 10.4, one can obtain a similar characterization for the space of actions.

Theorem 12.7. *Let Γ be an infinite countable group. Then Γ does not have* HAP *iff* $\overline{\mathrm{MIX}(\Gamma, X, \mu)} \subseteq \mathrm{ERG}(\Gamma, X, \mu)$ (*where closure is in the weak topology*).

Proof. $\Rightarrow$: If Γ does not have HAP, then

$$\mathrm{MIX}(\Gamma, X, \mu) \subseteq \{a \in A(\Gamma, X, \mu) : 1_\Gamma \not\prec \kappa_0^a\}.$$

Assume now, towards a contradiction, that $a_n \in \mathrm{MIX}(\Gamma, X, \mu)$ and $a_n \to a \notin \mathrm{ERG}(\Gamma, X, \mu)$. Let $b = \prod_n a_n$. Then, by 10.4, $a_n \prec b, \forall n$, so $a \prec b$ and thus $\kappa_0^a \prec \kappa_0^b$. Now, as a is not ergodic, $1_\Gamma \leq \kappa_0^a$, so $1_\Gamma \prec \kappa_0^b$, contradicting the fact that b is mixing (as a product of mixing actions.)

$\Leftarrow$: Now assume that Γ has HAP. Then, by 11.1, there is an action $a_0 \in \mathrm{MIX}(\Gamma, X, \mu) \setminus \mathrm{E_0RG}(\Gamma, X, \mu)$ and thus, for the action a_0, there is an almost invariant sequence of Borel sets $\{A_n\}$ in X with $\mu(A_n) = \frac{1}{2}$. Then, as in the proof of 12.2, ii), we see that $b = \frac{1}{2}a_0 + \frac{1}{2}a_0 \prec a_0 \times a_0$. Clearly b is not ergodic, $a_0 \times a_0$ is mixing, and so $b \in \overline{\mathrm{MIX}(\Gamma, X, \mu)}$, thus $\overline{\mathrm{MIX}(\Gamma, X, \mu)} \not\subseteq \mathrm{ERG}(\Gamma, X, \mu)$. □

Note also that a similar argument shows that if Γ has HAP, then the space $\overline{\mathrm{MIX}(\Gamma, X, \mu)}$ is closed under convex combinations. Indeed let

$$a_1, a_2 \in \overline{\mathrm{MIX}(\Gamma, X, \mu)}, 0 \leq \lambda_i \leq 1, \lambda_1 + \lambda_2 = 1,$$

and let $a_i^{(n)} \in \mathrm{MIX}(\Gamma, X, \mu), a_i^{(n)} \to a_i$ (in the weak topology). Then if $a^{(n)} = a_1^{(n)} \times a_2^{(n)}, \lambda_1 a_1^{(n)} + \lambda_2 a_2^{(n)} \prec \lambda_1 a^{(n)} + \lambda_2 a^{(n)} \prec a_0 \times a^{(n)}$, for an appropriate $a_0 \in \mathrm{MIX}(\Gamma, X, \mu) \setminus \mathrm{E_0RG}(\Gamma, X, \mu)$, and, as $a_0, a^{(n)}, a_0 \times a^{(n)} \in \mathrm{MIX}(\Gamma, X, \mu)$, we have $\lambda_1 a_1 + \lambda_2 a_2 \in \overline{\mathrm{MIX}(\Gamma, X, \mu)}$. Thus an infinite Γ has HAP iff $\overline{\mathrm{MIX}(\Gamma, X, \mu)}$ is closed under convex combinations. (For the analogous result concerning shift-invariant measures on $\mathbb{T}^\Gamma$, see Glasner [Gl2], 13.21.)

We also have the following characterization of property (T) groups.

Theorem 12.8. *Let Γ be an infinite countable group. Then Γ has property* (T) *iff* $\overline{\mathrm{WMIX}(\Gamma, X, \mu)} \subseteq \mathrm{ERG}(\Gamma, X, \mu)$ (*where closure is in the weak topology*) *iff* $\mathrm{WMIX}(\Gamma, X, \mu)$ *is closed in the weak topology.*

Proof. The first equivalence is proved as in 12.7, using also 12.2. If Γ has property (T), then $\mathrm{ERG}(\Gamma, X, \mu)$ is closed. But

$$a \in \mathrm{WMIX}(\Gamma, X, \mu) \Leftrightarrow a \times a \in \mathrm{ERG}(\Gamma, X \times X, \mu \times \mu)$$

and $a \mapsto a \times a$ is continuous in the weak topology, so $\overline{\mathrm{WMIX}(\Gamma, X, \mu)}$ is also closed. Clearly if $\mathrm{WMIX}(\Gamma, X, \mu)$ is weakly closed, $\overline{\mathrm{WMIX}(\Gamma, X, \mu)} = \mathrm{WMIX}(\Gamma, X, \mu) \subseteq \mathrm{ERG}(\Gamma, X, \mu)$. $\square$

Again we also have that Γ does not have property (T) iff $\overline{\mathrm{WMIX}(\Gamma, X, \mu)}$ is closed under convex combinations.

In the recent paper Kerr-Pichot [KP] the authors independently note that if Γ has property (T), then $\mathrm{WMIX}(\Gamma, X)$ is closed in the weak topology. Moreover they establish the following strengthening of 12.2, (ii), which answers a question of Bergelson-Rosenblatt [BR], p. 80.

Theorem 12.9 (Kerr-Pichot [KP]). *If a countable group* Γ *does not have property* (T)*, then* $\mathrm{WMIX}(\Gamma, X, \mu)$ *is dense* G_δ *in the weak topology of* $A(\Gamma, X, \mu)$.

Proof (sketch). Fix a sequence of Borel finite partitions $\mathcal{P}_1, \mathcal{P}_2, \dots$ of X, each refining the preceding one, such that the union of the finite Boolean algebras generated by each $\mathcal{P}_n$ is dense in the measure algebra. Then the set of weak mixing actions is equal to the intersection of the sets

$$W_{n,m} = \{a \in A(\Gamma, X, \mu) : \exists\gamma\forall A, B \in \mathcal{P}_n(|\mu(\gamma^a(A) \cap B) - \mu(A)\mu(B)| < \frac{1}{m})\}.$$

So, by the Baire Category Theorem, it is enough to show that for each finite Borel partition $\mathcal{P}$ of X and $\epsilon > 0$, the open set

$$\{a \in A(\Gamma, X, \mu) : \exists\gamma\forall A, B \in \mathcal{P}(|\mu(\gamma^a(A) \cap B) - \mu(A)\mu(B)| < \epsilon)\}$$

is dense in the weak topology. By considering refinements of partitions, it is enough to show that given $a \in A(\Gamma, X, \mu)$, a finite Borel partition $\mathcal{Q}$ of $X, 1 \in F \subseteq \Gamma$ finite and $\delta > 0$, there is $b \in A(\Gamma, X, \mu)$ such that $\mu(\gamma^b(A)\Delta\gamma^a(A)) < \delta, \forall A \in \mathcal{Q}, \forall\gamma \in F$ and $\exists\gamma(|\mu(\gamma^b(A) \cap B) - \mu(A)\mu(B)| < \delta, \forall A, B \in \mathcal{Q})$.

Since Γ does not have property (T), 11.2 shows that $\mathrm{WMIX}(\Gamma, X, \mu) \setminus \mathrm{E_0RG}(\Gamma, X, \mu) \neq \emptyset$ and so we can find $c \in \mathrm{WMIX}(\Gamma, X, \mu)$ such that for each $\epsilon > 0$, finite $F \subseteq \Gamma$, there is a set A with $\mu(A) = 1/2$ and $\mu(\gamma^c(A)\Delta A) < \epsilon, \forall\gamma \in F$ (see, e.g., Hjorth-Kechris [HK3], A2.2). By considering the weak mixing product c^n and the partition $\{\prod_{1\leq i\leq n} A^{p(i)} : p \in \{-1, 1\}^n\}$, where $A^1 = A, A^{-1} = X \setminus A$, it follows that for each $\epsilon > 0, n \geq 1, F \subseteq \Gamma$ finite, we can find $d \in \mathrm{WMIX}(\Gamma, X, \mu)$ and a Borel partition $\mathcal{R}$ of X into 2^n pieces of equal measure such that $\mu(\gamma^d(A)\Delta A) < \epsilon, \forall\gamma \in F, \forall A \in \mathcal{R}$. (Alternatively, one can use the argument in the second part of the proof of 10.6 to show that we can actually take $d = c$.)

Fix now $a \in A(\Gamma, X, \mu), 1 \in F \subseteq \Gamma$ finite, a finite Borel partition $\mathcal{Q}$ of X and $\delta > 0$, in order to find $b \in A(\Gamma, X, \mu)$ such that $\mu(\gamma^b(A)\Delta\gamma^a(A)) < \delta, \forall A \in \mathcal{Q}, \forall\gamma \in F$ and $\exists\gamma[|\mu(\gamma^a(A) \cap B) - \mu(A)\mu(B)| < \delta, \forall A, B \in \mathcal{Q}]$.

First let n, ϵ be such that $2^{-n} < \delta, 2^n\epsilon < \delta$. Then find $d, \mathcal{R}$ for these ϵ, n, F. Say $\mathcal{R} = \{R_1, \dots, R_m\}$, with $m = 2^n$. Consider the product space

(X^{m+1}, μ^{m+1}) and for $B \subseteq X$, let

$$\hat{B} = \bigcup_{i=1}^{m} (R_i \times X \times \cdots \times B \times \cdots \times X),$$

where B appears in the $(i+1)$th coordinate in $R_i \times X \times \cdots \times B \times \cdots \times X$. Clearly $\mu^{m+1}(\hat{B}) = \mu(B)$, and $B \mapsto \hat{B}$ preserves intersections and unions.

If $\hat{a}$ is the action $i_\Gamma \times a^m$, where i_Γ is the trivial action of Γ on X, then $\widehat{\gamma^a(B)} = \gamma^{\hat{a}}(\hat{B})$, for any Borel set $B \subseteq X$ and $\gamma \in \Gamma$. It follows that there is a isomorphism $\varphi : (X^{m+1}, \mu^{m+1}) \to (X, \mu)$ with $\varphi(\gamma^{\hat{a}}(\hat{A})) = \gamma^a(A), \forall \gamma \in F, A \in \mathcal{Q}$. Let now $e = d \times a^m$ and b be the isomorphic copy of e on (X, μ) induced by φ. Then this b works. □

Corollary 12.10. *A countable group* Γ *does not have property* (T) *iff* $\mathrm{ERG}(\Gamma, X, \mu)$ *is dense in the weak topology of* $A(\Gamma, X, \mu)$ *iff* $\mathrm{WMIX}(\Gamma, X, \mu)$ *is dense in the weak topology of* $A(\Gamma, X, \mu)$.

From 12.7 it follows that for every Γ that does not have HAP, the set $\mathrm{MIX}(\Gamma, X, \mu)$ is not dense in $A(\Gamma, X, \mu)$. In an earlier version of this work it was asked whether for any infinite group with HAP, $\mathrm{MIX}(\Gamma, X, \mu)$ is dense in $A(\Gamma, X, \mu)$. This has now been answered by Hjorth.

Theorem 12.11 (Hjorth [Hj4]). *An infinite countable group* Γ *has the* HAP *iff* $\mathrm{MIX}(\Gamma, X, \mu)$ *is dense in the weak topology of* $A(\Gamma, X, \mu)$.

I do not know if there are Γ for which $\mathrm{MIX}(\Gamma, X, \mu)$ is comeager in the weak topology of $A(\Gamma, X, \mu)$.

13. Turbulence of conjugacy in the ergodic actions

(A) We will now consider the conjugacy action of $\mathrm{Aut}(X, \mu)$ on the space $\mathrm{ERG}(\Gamma, X, \mu)$. First we show that this action has a dense orbit.

Theorem 13.1. *For any infinite countable group* Γ*, there is a free action in the space* $\mathrm{ERG}(\Gamma, X, \mu)$ *with dense conjugacy class in* $(\mathrm{ERG}(\Gamma, X, \mu), w)$. *In particular the set of free actions in* $\mathrm{ERG}(\Gamma, X, \mu)$ *is dense* G_δ *in the space* $(\mathrm{ERG}(\Gamma, X, \mu), w)$.

Proof. The proof is a modification of that of 10.7. Fix a dense set $\{a_n\}$ in $(\mathrm{ERG}(\Gamma, X, \mu), w)$, including a free action. For each n also fix a set $X_n \subseteq X$ with $\mu(X_n) = 1$, such that X_n is a_n-invariant and μ is the unique ergodic, invariant measure for the action a_n restricted to X_n. Let $a = \prod_n a_n$. By the ergodic decomposition there is an ergodic, invariant measure ρ for a with $\rho(\prod_n X_n) = 1$ and a is free ρ-a.e. Then if $\pi_n : X^{\mathbb{N}} \to X$ denotes the nth projection, $\pi_n(\{x_i\}) = x_n$, clearly $(\pi_n)_* \rho = \mu$. So ρ is a joining of $\{a_n\}$. Then, by 10.4, $a_n \prec (\prod_n a_n)_\rho$ and $(\prod_n a_n)_\rho$ is free. □

It is clear that for any group Γ, if $a \in \mathrm{ERG}(\Gamma, X, \mu)$ has dense conjugacy class in the weak topology of the space of ergodic actions, then a is free (since if a is not free, then for some $\gamma \in \Gamma \setminus \{1\}, \epsilon < 1, \delta_u(\gamma^a, 1) \leq \epsilon$ and closed balls in δ_u are weakly closed). Foreman-Weiss [FW] show that if Γ is amenable, then the conjugacy class of *every* free ergodic action is dense in $(\mathrm{FRERG}(\Gamma, X, \mu), u)$ and thus it is dense in $(\mathrm{ERG}(\Gamma, X, \mu), w)$. However, as we will now see in 13.2 below, this first property can only hold for amenable Γ. I do not know a characterization of the class of free ergodic actions that have weakly dense conjugacy class in $\mathrm{ERG}(\Gamma, X, \mu)$ (for non-amenable Γ).

Proposition 13.2. *Let Γ be an infinite countable group. Let s_Γ be the shift action of Γ on (X, μ), where $X = 2^\Gamma$ and μ is the usual product measure. Then the following are equivalent:*

(i) Γ is amenable.

(ii) Every action in $\mathrm{FRERG}(\Gamma, X, \mu)$ has dense conjugacy class in the space $(A(\Gamma, X, \mu), w)$.

(iii) s_Γ has dense conjugacy class in $(A(\Gamma, X, \mu), w)$, i.e., $a \prec s_\Gamma$ for every $a \in A(\Gamma, X, \mu)$.

(iv) $i_\Gamma \prec s_\Gamma$.

(v) There is an action in $\mathrm{FRERG}(\Gamma, X, \mu)$ with dense conjugacy class in the space $(\mathrm{FRERG}(\Gamma, X, \mu), u)$.

(vi) Every action in $\mathrm{FRERG}(\Gamma, X, \mu)$ has dense conjugacy class in the space $(\mathrm{FRERG}(\Gamma, X, \mu), u)$.

Proof. Let $s = s_\Gamma$. (i) $\Rightarrow$ (vi) is the aforementioned result of Foreman-Weiss [FW], and clearly (vi) $\Rightarrow$ (v). Next we claim that (v) $\Rightarrow$ (vi). To see this, we note that the conjugacy action of $\mathrm{Aut}(\Gamma, X, \mu)$ on $A(\Gamma, X, \mu)$ is an action by isometries for the metric $\delta_{\Gamma,u}$. From this it is easy to see that the uniform closures of the conjugacy classes in the space $A(\Gamma, X, \mu)$ form a partition of this space (see, e.g., the proof of 23.1 below). So if some conjugacy class is dense in $(\mathrm{FRERG}(\Gamma, X, \mu), u)$, every conjugacy class is dense in this space.

We now show that (vi) $\Rightarrow$ (ii). Indeed, if (vi) holds, then Γ cannot have property (T) by 14.2 and the fact that not all free ergodic actions of Γ are conjugate, which follows, for instance, from 13.7. But then by 13.1 and 12.2, ii), we obtain immediately (ii).

Clearly (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Finally to see that (iv) $\Rightarrow$ (i) note that $\kappa_0^{i_\Gamma} = \infty \cdot 1_\Gamma$ and $\kappa_0^s \prec \lambda_\Gamma$ (see Bekka-de la Harpe-Valette [BdlHV], E.3.5), so if $i_\Gamma \prec s_\Gamma$, we have $1_\Gamma \prec \lambda_\Gamma$ and thus Γ is amenable (see, e.g., Zimmer [Zi1] or Bekka-de la-Harpe-Valette [BdlHV], G.3.2). □

Remark. I would like to thank Eli Glasner for pointing out an inaccuracy in an earlier version of this proposition.

Remark. In Ageev [A3] it is shown that if an infinite Γ is amenable, then every action in $\mathrm{FR}(\Gamma, X, \mu)$ has dense conjugacy class in the space $(A(\Gamma, X, \mu), w)$, thus, using also 13.2, Γ is amenable iff every action in $\mathrm{FR}(\Gamma, X, \mu)$ has dense conjugacy class in $(A(\Gamma, X, \mu), w)$. This answers a question in the above paper [A3].

Remark. Note that by 12.2 and 13.1, Γ does not have property (T) iff there is an action in $\text{FRERG}(\Gamma, X, \mu)$ with dense conjugacy class in the space $(A(\Gamma, X, \mu), w)$.

Remark. One can also ask if the weakening of (iii), where we require only that $a \prec s_\Gamma$ for every $a \in \text{ERG}(\Gamma, X, \mu)$, is enough to imply that Γ is amenable. I do not know the answer but one can see that this weakening implies that Γ does not contain F_2. To prove this, we consider cases: If Γ does not have property (T), then, by 12.2, ii), $a \prec s_\Gamma$ for all $a \in A(\Gamma, X, \mu)$, so Γ is amenable. So we can assume that Γ has property (T). We will then show that every infinite index subgroup $\Delta \leq \Gamma$ must be amenable. This implies that $F_2 \leq \Gamma$ fails, since otherwise F_2 would have finite index and thus property (T) (see Bekka-de la Harpe-Valette [BdlHV], 1.7.1), which is of course a contradiction.

So fix $\Delta \leq \Gamma$ of infinite index, let $X = \Gamma/\Delta$ and consider the translation action of Γ on $X : \gamma \cdot \delta\Delta = \gamma\delta\Delta$. Then let a be the shift action of Γ on $2^X : a(\gamma, f)(x) = f(\gamma^{-1} \cdot x)$, for $\gamma \in \Gamma, f \in 2^X, x \in X$. One can identify 2^X with the compact Cantor group $\mathbb{Z}_2^X$, so this is an action by automorphisms on this group. Using 4.3 of Kechris [Kec4], we can see that (since X is infinite) a is ergodic, so $a \prec s_\Gamma$ and therefore $\kappa_0^a \prec \kappa_0^{s_\Gamma} \prec \lambda_\Gamma$, i.e., the action a is tempered, in the terminology of [Kec4]. Using then 4.6 of [Kec4], we see that the stabilizer of any point in the action of Γ on X is amenable, i.e., Δ is amenable.

To summarize, we have seen that if Γ is such that it satisfies $\forall a \in \text{ERG}(\Gamma, X, \mu)(a \prec s_\Gamma)$ but Γ is not amenable, then Γ must have property (T) and every infinite index subgroup of Γ must be amenable, so that $F_2 \not\leq \Gamma$. We also note that it must have the following property: Γ is minimally almost periodic, i.e., has no non-trivial finite-dimensional unitary representations.

Indeed, this property is equivalent to the property that $\text{WMIX}(\Gamma, X, \mu) = \text{ERG}(\Gamma, X, \mu)$ (see Schmidt [Sc4] and Section 11, **(C)**). If this failed for a Γ as above, then there is $a \in \text{ERG}(\Gamma, X, \mu) \setminus \text{WMIX}(\Gamma, X, \mu)$, so $a \prec s_\Gamma$ and thus $\kappa_0^a \prec \kappa_0^{s_\Gamma} \prec \lambda_\Gamma$ and κ_0^a is not weak mixing, so it has a non-0 finite-dimensional subrepresentation π. Thus $\pi \prec \lambda_\Gamma$, i.e., λ_Γ weakly contains a non-0 finite-dimensional representation, which implies that Γ is amenable (see Dixmier [Di], 18.9.5 and 18.3.6), a contradiction.

I do not know whether property (T) groups with the above properties exist.

Finally, I also do not know if the following weakening of 13.2, (ii) is enough to guarantee amenability of Γ: Every action in $\text{FRERG}(\Gamma, X, \mu)$ has a dense conjugacy class in $(\text{ERG}(\Gamma, X, \mu), w)$.

Remark. For each $T \in \text{ERG}$ we consider the corresponding unitary operator $U_T^0 \in U(L_0^2(X, \mu))$, induced by T on $L_0^2(X, \mu)$ and the associated set $M_T^0 \subseteq \{1, 2, \dots\} \cup \{\infty\}$ of essential values of its multiplicity function, i.e., the set of numbers $1 \leq n \leq \infty$ for which the multiplicity function of U_T^0 obtains the value n at a set of positive measure with respect to the maximal spectral type of U_T^0. It is a major open problem in the spectral theory of ergodic transformations to characterize the sets M_T^0 corresponding to $T \in$ ERG and it is indeed open whether *any* $A \subseteq \{1, 2, \dots\} \cup \{\infty\}$ is of the form M_T^0 (see Katok-Thouvenot [KTh], 3.6). It is of course clear that the map $T \in \mathrm{ERG} \mapsto M_T^0 \in 2^{\bar{\mathbb{N}}}$, where $\bar{\mathbb{N}} = \{1, 2, \dots\} \cup \{\infty\}$ and we identify subsets of $\bar{\mathbb{N}}$ with their characteristic functions, is Borel, so $\{M_T^0 : T \in \mathrm{ERG}\}$ is analytic but it is not clear whether it is Borel or not.

An old problem, known as Rokhlin's Problem, asked whether the sets $\{n\}, n = 2, 3, 4, \dots$ are realized as M_T^0 for ergodic T. An affirmative answer was given recently by Ageev [A2] (and Ryzhikov [Ry2] for $n = 2$). Ageev's method is to find for each n, a pair (Γ_n, γ_n) of a countable group Γ_n and an element $\gamma_n \in \Gamma_n$ such that for the generic $a \in A(\Gamma_n, X, \mu)$, if $T = \gamma_n^a$, then $M_T^0 = \{n\}$. Ageev suggests that there should be a *spectral rigidity*, i.e., for any pair (Γ, γ) of a countable group and $\gamma \in \Gamma, M_{\gamma^a}^0$ is fixed for the generic $a \in A(\Gamma, X, \mu)$ (with respect to the weak topology w). He shows that $\sup(M_{\gamma^a}^0)$ is indeed fixed for the generic a, provided $(A(\Gamma, X, \mu), w)$ has the *Rokhlin property*, i.e., admits a dense conjugacy class. Of course 10.7 asserts that this is indeed the case for all Γ, so that we can in fact infer (given that the map $a \mapsto M_{\gamma^a}^0$ is Borel and invariant under conjugacy, and using, e.g., Kechris [Kec2], 8.46) that we have full spectral rigidity, i.e., for any pair (Γ, γ), $M_{\gamma^a}^0$ is fixed for the generic $a \in A(\Gamma, X, \mu)$. (A similar remark for $\sup(M_{\gamma^a}^o)$ is also made in Glasner-Thouvenot-Weiss [GTW]. A recent preprint of Ageev [A3] also contains this fact for $M_{\gamma^a}^0$.) Of course γ^a will not be ergodic if a is not ergodic, so it may be more reasonable to look instead at the Polish space $(\mathrm{ERG}(\Gamma, X, \mu), w)$. By 12.2, $\mathrm{ERG}(\Gamma, X, \mu)$ is meager in $(A(\Gamma, X, \mu), w)$, if Γ has property (T), and dense G_δ if Γ does not have property (T), so it is irrelevant whether we use $A(\Gamma, X, \mu)$ or $\mathrm{ERG}(X, \Gamma, \mu)$ if Γ does not have property (T). In any case, even if we work within $\mathrm{ERG}(\Gamma, X, \mu)$, 13.1 implies that we still have a dense conjugacy class and thus $M_{\gamma^a}^0$ is fixed for the generic $a \in \mathrm{ERG}(\Gamma, X, \mu)$, for each infinite Γ.

(B) We next study (generic) turbulence of the conjugacy action of the group $\mathrm{Aut}(X, \mu)$ on $\mathrm{ERG}(\Gamma, X, \mu)$.

Theorem 13.3 (Kechris). *Suppose a countable group* Γ *does not have property* (T). *Then the following are equivalent:*

(i) Every conjugacy class in $\mathrm{ERG}(\Gamma, X, \mu)$ *is meager in the topological space* $(\mathrm{ERG}(\Gamma, X, \mu), w)$.

(ii) The conjugacy action of $(\mathrm{Aut}(X, \mu), w)$ *on* $(\mathrm{ERG}(\Gamma, X, \mu), w)$ *is generically turbulent.*

Proof. It is of course enough to show that (i) $\Rightarrow$ (ii). We work below in the weak topology.

By 10.7, 12.2 (ii) and 12.3, there is an $a \in \mathrm{ERG}(\Gamma, X, \mu)$ with dense conjugacy class which is not E_0-ergodic. It is enough to show that a is a turbulent point. Let $E = E_a$ be the equivalence relation induced by a. Let for $\epsilon > 0, \gamma_1, \dots, \gamma_k \in \Gamma, A_1, \dots, A_\ell \in \text{MALG}_\mu$,

$$U = \{b \in \text{ERG}(\Gamma, X, \mu) : \forall i \leq k \forall j \leq \ell [\mu(\gamma_i^b(A_j) \Delta \gamma_i^a(A_j)) < \epsilon]\}.$$

Fix also an open nbhd V of $1 \in \text{Aut}(X, \mu)$. We will show that the local orbit $\mathcal{O}(a, U, V)$ is dense in U. Since the conjugacy glass of a is dense in $\text{ERG}(\Gamma, X, \mu)$ it is enough to show that every $TaT^{-1} \in U$ can be weakly approximated by elements of $\mathcal{O}(a, U, V)$. Since E is ergodic, $[E]$ is weakly dense in $\text{Aut}(X, \mu)$, so it is enough to take $TaT^{-1} \in U$ with $T \in [E]$. Finally, as the periodic elements are uniformly dense in $[E]$, we can assume that T is periodic. We can now proceed as in the proofs of 5.1, 5.2, using the fact that each γ^a can be uniformly approximated by periodic elements of $[E]$.

More precisely, let $\pi : E \to E_0$ be a Borel homomorphism with the preimage of every E_0-class null. Let $E_0 = \bigcup_{n=1}^\infty E_n, E_1 \subseteq E_2 \subseteq \dots$ finite Borel. Choose also $\delta > 0$ sufficiently small (as needed below). Then choose periodic $\gamma'_1, \dots, \gamma'_k \in [E]$, so that $\delta'_u(\gamma_i^a, \gamma'_i) < \delta/2, \forall i \leq k$, and, as in the proof of 5.2, choose N large enough so that there are $\bar{T}, \bar{\gamma}_1, \dots, \bar{\gamma}_k \in [\pi^{-1}(E_N) \cap E]$ with $\delta'_u(T, \bar{T}) < \delta, \delta'_u(\gamma'_i, \bar{\gamma}_i) < \delta/2, \forall i \leq k$. Then, again as in the proof of 5.2, we can find a continuous $\lambda \mapsto X_\lambda \in \text{MALG}_\mu, \lambda \in [0,1], X_0 = \emptyset, X_1 = X, 0 \leq \lambda \leq \lambda' \Rightarrow X_\lambda \subseteq X_{\lambda'}, \mu(X_\lambda \setminus X_{\lambda'}) \leq \lambda' - \lambda$, and X_λ $\pi^{-1}(E_N)$-invariant, so that it is also $\bar{T}, \bar{\gamma}_i$ $(1 \leq i \leq k)$-invariant. Put $T_\lambda = \bar{T}|X_\lambda \cup \text{id}|(X \setminus X_\lambda)$, so that X_λ is also $T_\lambda, \bar{\gamma}_i$ $(1 \leq i \leq k)$-invariant. Since $T_0 = 1, T_1 = \bar{T}$, as in the proof of 5.1, we have that for $A \in \text{MALG}_\mu, 1 \leq i \leq k$,

$$T_\lambda \bar{\gamma}_i T_\lambda^{-1}(A) \Delta \bar{\gamma}_i(A) \subseteq (\bar{T} \bar{\gamma}_i \bar{T}^{-1}(A) \Delta \bar{\gamma}_i(A)) \cap X_\lambda.$$

It follows that

$$\begin{aligned} \mu(T_\lambda \gamma_i^a T_\lambda^{-1}(A) \Delta \gamma_i^a(A)) &\leq \mu(T_\lambda \gamma_i^a T_\lambda^{-1}(A) \Delta \bar{\gamma}_i(A)) + \delta \\ &\leq \mu(T_\lambda \bar{\gamma}_i T_\lambda^{-1}(A) \Delta \bar{\gamma}_i(A)) + 2\delta \\ &\leq \mu(\bar{T} \bar{\gamma}_i \bar{T}^{-1}(A) \Delta \bar{\gamma}_i(A)) + 2\delta \\ &\leq \mu(T \gamma_i^a T^{-1}(A) \Delta \gamma_i^a(A)) + 6\delta, \end{aligned}$$

so if δ is small enough, clearly $T_\lambda a T_\lambda^{-1} \in U, \forall \lambda \in [0,1]$. Moreover, for any $\gamma \in \Gamma$,

$$\delta'_u(T_1 \gamma^a T_1^{-1}, T \gamma^a T^{-1}) = \delta'_u(\bar{T} \gamma^a \bar{T}^{-1}, T \gamma^a T^{-1}) \leq 2\delta,$$

and this completes the proof. □

Foreman-Weiss [FW], using ideas about entropy of actions of amenable groups, show that every conjugacy class of an action of an amenable group is meager, thus we have the following result.

Theorem 13.4 (Foreman-Weiss). *If Γ is an infinite amenable group, the conjugacy action of the group $(\text{Aut}(X, \mu), w)$ on $(\text{ERG}(\Gamma, X, \mu), w)$ is generically turbulent.*

It is also clear that the conjugacy classes of actions of free groups are meager, so we have the next fact.

Theorem 13.5 (Kechris). *If Γ is a free group, then the conjugacy action of the group* $(\mathrm{Aut}(X,\mu),w)$ *on* $(\mathrm{ERG}(\Gamma,X,\mu),w)$ *is generically turbulent.*

In fact, more generally, if Γ,Δ are countable groups without property (T) and the conjugacy action of $(\mathrm{Aut}(X,\mu),w)$ on each one of the spaces $(\mathrm{ERG}(\Gamma,X,\mu),w)$, $(\mathrm{ERG}(\Delta,X,\mu),w)$ is generically turbulent, so in the action of $(\mathrm{Aut}(X,\mu),w)$ on $(\mathrm{ERG}(\Gamma*\Delta,X,\mu),w)$ (note that $\Gamma*\Delta$ does not have property (T) as well.)

The following problem remains open.

Problem 13.6. *For which infinite countable groups Γ are the conjugacy classes in* $\mathrm{ERG}(\Gamma,X,\mu)$ *meager in the weak topology of* $\mathrm{ERG}(\Gamma,X,\mu)$*? Is 13.3 true for property* (T) *groups?*

In the next section we will see an important fact concerning conjugacy classes in $\mathrm{ERG}(\Gamma,X,\mu)$ for property (T) groups.

(C) Although the question of turbulence for the conjugacy action of $\mathrm{Aut}(X,\mu)$ on $\mathrm{ERG}(\Gamma,X,\mu)$ is still unresolved for arbitrary Γ, one can still derive from 13.4 and work of Hjorth [Hj3] that the conjugacy equivalence relation on $\mathrm{ERG}(\Gamma,X,\mu)$ cannot be classified by countable structures.

Let $U(H)$ be the unitary group of an infinite-dimensional separable Hilbert space H with the weak (equivalently the strong) topology. This is a Polish group. For each countable group Γ, let $\mathrm{Rep}(\Gamma,H)$ be the space of unitary representations of Γ on H, i.e., the closed subspace of H^Γ consisting of all homomorphisms of Γ on H (see Appendix H). Finally, let $\mathrm{Irr}(\Gamma,H)$ be the subset of $\mathrm{Rep}(\Gamma,H)$ consisting of all irreducible representations. This is a G_δ subset of $\mathrm{Rep}(\Gamma,H)$, hence Polish. The group $U(H)$ acts continuously on $\mathrm{Rep}(\Gamma,H)$ by conjugation. The corresponding equivalence relation is *isomorphism* or *unitary equivalence* of representations, in symbols $\pi\cong\rho$. We finally put $\pi\leq\rho$ if π is isomorphic to a *subrepresentation* of ρ.

Hjorth [Hj3] has shown that there is a conjugacy invariant G_δ set

$$G_{\Gamma,H}\subseteq\mathrm{Irr}(\Gamma,H)$$

such that the action of $U(H)$ on $G_{\Gamma,H}$ is turbulent, provided that Γ is not abelian-by-finite. This strengthens the result of Thoma [Th], as it implies that unitary equivalence of irreducible unitary representations cannot be classified by countable structures. (However Thoma's result is used in Hjorth's proof.) See Appendix H, (**C**) for an exposition of Hjorth's work.

It is known (see Appendix E) that one can assign in a Borel way to each $\pi\in\mathrm{Rep}(\Gamma,H)$ an action $a_\pi\in A(\Gamma,X,\mu)$, so that

$$\pi\cong\rho\Rightarrow a_\pi\cong a_\rho,$$

and moreover if $\kappa_0^{a_\pi}$ is the Koopman representation on $L^2_0(X,\mu)$ associated with a_π, then

$$\pi\leq\kappa_0^{a_\pi}.$$

We now have the following result.

Theorem 13.7 (Foreman-Weiss, Hjorth). *Let Γ be an infinite countable group. Then conjugacy of measure preserving, free, ergodic actions of Γ cannot be classified by countable structures.*

Proof. If Γ is abelian-by-finite, this follows from 13.4.

So assume Γ is not abelian-by-finite, and assume, towards a contradiction, that there is a countable language L, so that, denoting by X_L the space of countable structures for L and by $\cong$ the isomorphism relation on X_L, there is a Borel function $F : \mathrm{FRERG}(\Gamma, X, \mu) \to X_L$ with

$$a \cong b \Leftrightarrow F(a) \cong F(b).$$

Now if $\pi \in G_{\Gamma,H} \subseteq \mathrm{Irr}(\Gamma, H), a_\pi$ is weak mixing, by the remarks following E.1, and so if a_0 is any fixed action in $\mathrm{FRERG}(\Gamma, X, \mu), a_0 \times a_\pi$ is in $\mathrm{FRERG}(\Gamma, X, \mu)$. Thus, for $\pi, \rho \in G_{\Gamma,H} \subseteq \mathrm{Irr}(\Gamma, H)$,

$$\pi \cong \rho \Rightarrow F(a_0 \times a_\pi) \cong F(a_0 \times a_\rho).$$

Since the conjugacy action on $G_{\Gamma,H}$ is turbulent, there is some $M_0 \in X_L$ with

$$F(a_0 \times a_\pi) \cong M_0$$

for comeager many $\pi \in G_{\Gamma,H}$. So there is $a \in \mathrm{FRERG}(\Gamma, X, \mu)$, for which on a comeager set of $\pi \in G_{\Gamma,H}, a_0 \times a_\pi \cong a$. Since every conjugacy class in $G_{\Gamma,H}$ is meager in $G_{\Gamma,H}$, there are uncountably many pairwise non-conjugate $\pi \in \mathrm{Irr}(\Gamma, H)$ such that $a_0 \times a_\pi \cong a$, and thus $\pi \leq \kappa_0^{a_\pi} \leq \kappa_0^{a_0 \times a_\pi} \cong \kappa_0^a$, which is a contradiction, as κ_0^a contains only countably many irreducible subrepresentations. □

Remark. In a recent preprint, Törnquist [To2] has extended 13.7 to the set of measure preserving, free, ergodic actions of Γ that can be extended to free actions of Δ, for any (fixed) $\Delta \geq \Gamma$.

Remark. By choosing in the preceding proof a_0 to be weak mixing, we see that 13.7 holds as well for measure preserving free, *weak mixing* actions of Γ. We also note that 13.7 and 2.2 of Hjorth-Kechris [HK1] imply that the equivalence relation E_0 can be Borel reduced to conjugacy of such actions. It has been shown recently by Ioana-Kechris-Tsankov-Epstein (see [IKT]) that these facts are true as well for measure preserving, free, *mixing* actions, provided that the group Γ is not amenable. I do not know if this holds for all infinite Γ.

(D) One can also consider unitary (spectral) equivalence in $A(\Gamma, X, \mu)$. Recall that two actions $a \in A(\Gamma, X, \mu), b \in A(\Gamma, Y, \nu)$ are called *unitarily* (or *spectrally*) *equivalent* if the corresponding Koopman representations κ^a, κ^b are isomorphic. We then have the following analog of 13.7.

Theorem 13.8 (Kechris). *Let Γ be an infinite countable group. Then unitary equivalence of measure preserving, free, ergodic actions of Γ cannot be classified by countable structures.*

Proof. We will again consider cases depending on whether Γ is abelian-by-finite or not.

If Γ is not abelian-by-finite, then the argument in the proof of 13.7 works as well for unitary equivalence instead of conjugacy.

Next consider the case where Γ is abelian-by-finite. Let $\Delta \lhd \Gamma, \Delta$ infinite abelian, $[\Gamma : \Delta] < \infty$. From now on we will use the notation and results of Appendices F, G. For $a \in \mathrm{FRERG}(\Delta, X, \mu)$, let $\mathrm{Ind}_\Delta^\Gamma(a) \in \mathrm{FRERG}(\Gamma, X \times T, \mu \times \nu_T)$ be the induced action. Clearly $a \mapsto f(a) = \mathrm{Ind}_\Delta^\Gamma(a)$ is Borel. Assume now, towards a contradiction, that there is a Borel map

$$F : \mathrm{FRERG}(\Gamma, X \times T, \mu \times \nu_T) \to X_L$$

with

$$\kappa^c \cong \kappa^d \Leftrightarrow F(c) \cong F(d).$$

Then for $a, b \in \mathrm{FRERG}(\Delta, X, \mu)$,

$$a \cong b \Rightarrow f(a) \cong f(b) \Rightarrow F(f(a)) \cong F(f(b)).$$

So, by 13.4 and 10.8, there is a comeager set $A \subseteq \mathrm{FRERG}(\Delta, X, \mu)$ such that for $a, b \in A, F(f(a)) \cong F(f(b))$ and thus

$$\kappa^{\mathrm{Ind}_\Delta^\Gamma(a)} \cong \kappa^{\mathrm{Ind}_\Delta^\Gamma(b)},$$

i.e.,

$$\kappa^{\mathrm{Ind}_\Delta^\Gamma(a)} \cong \mathrm{Ind}_\Delta^\Gamma(\kappa^a) \cong \mathrm{Ind}_\Delta^\Gamma(\kappa^b) \cong \kappa^{\mathrm{Ind}_\Delta^\Gamma(b)}$$

and therefore $\mathrm{Ind}_\Delta^\Gamma(\kappa^a)|\Delta \cong \mathrm{Ind}_\Delta^\Gamma(\kappa^b)|\Delta$. So if σ_a, σ_b are the maximal spectral types for κ^a, κ^b, resp., so that $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_a, \sum_{t\in T}(\hat{\theta}_t)_*\sigma_b$ are the maximal spectral types for $\mathrm{Ind}_\Delta^\Gamma(\kappa^a)|\Delta, \mathrm{Ind}_\Delta^\Gamma(\kappa^b)|\Delta$, resp., then we have

$$(1) \qquad a, b \in A \Rightarrow \sum_{t\in T}(\hat{\theta}_t)_*\sigma_a \sim \sum_{t\in T}(\hat{\theta}_t)_*\sigma_b.$$

We will next use some ideas from Choksi-Nadkarni [CN] (see also [Na, Ch. 8]). First recall that for each $\rho \in P(\hat{\Delta})$ = the space of probability measures on $\hat{\Delta}$, $\{\nu \in P(\hat{\Delta}) : \nu \perp \rho\}$ is a G_δ set in $P(\hat{\Delta})$ (see, e.g., Kechris-Sofronidis [KS]). Since $a \in A(\Delta, X, \mu) \mapsto \kappa_0^a \in \mathrm{Rep}(\Delta, L_0^2(X, \mu))$ is continuous, so that $a \mapsto \sigma_{\kappa_0^a}$ is also continuous, we have that $\{a \in A(\Delta, X, \mu) : \sigma_{\kappa_0^a} \perp \rho\}$ and $B_\rho = \{a \in A(\Delta, X, \mu) : \sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^a} \perp \rho\}$ are G_δ as well (we work here in the weak topology of $A(\Delta, X, \mu)$) and also conjugacy invariant. We will find a free ergodic action $c_0 \in A(\Delta, X, \mu)$ such that the maximal spectral type $\sigma_{\kappa_0^{c_0}}$ of $\kappa_0^{c_0}$ is supported by a countable set, and thus so is $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^{c_0}}$, and, as a consequence, $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^{c_0}} \perp \lambda$, where $\lambda = \eta_{\hat{\Delta}}$ is the Haar measure on $\hat{\Delta}$. Since the conjugacy class of c_0 in $A(\Delta, X, \mu)$ is dense, by 13.2, it follows that B_λ is comeager and thus there is $b_0 \in B_\lambda \cap A$, i.e., we have $b_0 \in A$, $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^{b_0}} = \sigma_0 \perp \lambda$. Let now s be the Gaussian shift on $\mathbb{R}^\Delta$ associated to φ_1 = the characteristic function of $\{1\} \subseteq \Delta$ (see Appendix D). Then (see Appendix F, **(C)**) the maximal spectral type for κ_0^s is λ and $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^s} \sim \sum_{t\in T}(\hat{\theta}_t)_*\lambda = |T|\lambda$ (as each $\hat{\theta}_t$ preserves λ). Thus $s \in B_{\sigma_0}$. Again the conjugacy class of s is dense in $A(\Delta, X, \mu)$, as s is free and ergodic, so B_{σ_0} is comeager. Thus there is $a_0 \in A \cap B_{\sigma_0}$, i.e., $a_0 \in A$ and $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^{a_0}} \perp \sum_{t\in T}(\hat{\theta}_t)_*\sigma_{\kappa_0^{b_0}}$ and therefore $\sum_{t\in T}(\hat{\theta}_t)_*\sigma_{a_0} \not\sim \sum_{t\in T}(\hat{\theta}_t)_*\sigma_{b_0}$, where $\sigma_{a_0} = \delta_1 + \sigma_{\kappa_0^{a_0}}$ is the maximal spectral type for κ^{a_0} and similarly for σ_{b_0} (where δ_1 is the Dirac measure at $1 \in \hat{\Delta}$). This contradicts (1).

Construction of (an isomorphic copy) of c_0. By the next to last paragraph of Section 9, we can assume that Δ is a dense subgroup of a compact Polish group G. We take c_0 to be the translation action of Δ on (G, η_G), where η_G is the Haar measure on G. Clearly c_0 is free and ergodic. Then, using the fact that $\{\chi : \chi \in \hat{G}\}$ is an orthonormal basis for $L^2(G, \eta_G)$ and $\mathbb{C}\chi$ is invariant under the translation action, it is easy to check that the maximal spectral type for $\kappa_0^{c_0}$ is supported by a countable set and the proof is complete. □

Finally we have the following result.

Theorem 13.9. *Let* Γ *be an infinite countable group. Then weak isomorphism of measure preserving, free, ergodic actions of* Γ *cannot be classified by countable structures.*

Proof. If Γ is not abelian-by-finite, the argument in the proof of 13.7 works as well. If Γ is abelian-by-finite, then the argument in the proof of 13.8 applies and the proof is complete. □

Remark. Again 13.8 and 13.9 hold as well for weak mixing actions.

It appears that weak isomorphism $\cong^w$ is of a different nature than isomorphism $\cong$ or unitary equivalence, which are both Borel reducible to equivalence relations induced by continuous actions of Polish groups. For example, the following is open.

Problem 13.10. *Let* Γ *be a countable infinite group. Let* E_1 *be the following equivalence relation on* $\mathbb{R}^{\mathbb{N}}$*:*

$$(x_n)E_1(y_n) \Leftrightarrow \exists n \forall m \geq n(x_m = y_m).$$

Can E_1 *be Borel reduced to* $\cong^w$ *on* $\mathrm{FRERG}(\Gamma, X, \mu)$*?*

An affirmative answer would imply that $\cong^w$ cannot be Borel reducible to equivalence relations induced by continuous actions of Polish groups.

14. Conjugacy in ergodic actions of property (T) groups

(A) We will now use a technique employed in Hjorth [Hj4] to study conjugacy classes of ergodic actions of property (T) groups. It traces its origins in separability arguments used in the context of operator algebras in Popa [Po1]. In essence, Hjorth's proof, from the point of view presented here, showed that the conjugacy classes in $\mathrm{ERG}(\Gamma, X, \mu)$ are clopen in $(\mathrm{ERG}(\Gamma, X, \mu), u)$. However a more careful analysis, which is encapsulated in Lemma 14.1 below, can be used to derive stronger conclusions, see Theorem 14.2.

Let $\Gamma = \{\gamma_n\}$ be a countable group and let

$$\delta_{\Gamma,u}(a, b) = \sum_n 2^{-n}\delta_u(\gamma_n^a, \gamma_n^b)$$

be the metric giving the uniform topology on $A(\Gamma, X, \mu)$. Also if E is a countable measure preserving equivalence relation on (X, μ), then letting φ vary over all partial Borel automorphisms $\varphi : A \to B$, where A, B are Borel sets, we put

$$[[E]] = \{\varphi : \varphi(x)Ex, \forall x \in A\}.$$

Note that if $\varphi \in [[E]]$, then φ is measure preserving on its domain.

Below for two equivalence relations E, F, we define their *join*, $E \vee F$, to be the smallest equivalence relation containing E and F.

Lemma 14.1. *Let Γ be a countable group with property (T). Then given $\delta > 0$, there is $\eta > 0$, such that for any $a, b \in A(\Gamma, X, \mu)$, if $\delta_{\Gamma,u}(a, b) < \eta$, then there is $\varphi : A \to B, \varphi \in [[E_a \vee E_b]]$, such that A is a-invariant, B is b-invariant, $\varphi(a|A)\varphi^{-1} = b|B$, and $\mu(A) > 1 - 16\delta^2$. Moreover, $\mu(\{x \in A : \varphi(x) \neq x\}) \leq 4\delta^2$.*

Proof. Put below $E = E_a \vee E_b$. The main idea, which comes from Hjorth [Hj4], is to consider the product action $a \times b$ on X^2. Notice that it leaves $E \subseteq X^2$ invariant and the σ-finite measure M on E invariant, where for Borel $A \subseteq E$, $M(A) = \int \mathrm{card}(A_x) d\mu(x)$. Thus it gives a unitary representation of Γ on $L^2(E, M)$, which, viewed as an action of Γ on $L^2(E, M)$, is defined by

$$\gamma \cdot f(x, y) = f((\gamma^{-1})^a(x), (\gamma^{-1})^b(y)).$$

Claim. *If $\Delta : E \to \mathbb{C}$ is the diagonal function, $\Delta(x, y) = 1$ if $x = y$, $\Delta(x, y) = 0$, if $x \neq y$, then*

$$\|\gamma^{-1} \cdot \Delta - \Delta\|_2^2 = 2\delta_u(\gamma^a, \gamma^b)$$

Proof of the claim. We have

$$\begin{aligned} \|\gamma^{-1} \cdot \Delta - \Delta\|_2^2 &= \textstyle\int_E |\Delta(\gamma^a(x), \gamma^b(y)) - \Delta(x, y)|^2 dM(x, y) \\ &= \textstyle\int_{x=y} |\Delta(\gamma^a(x), \gamma^b(y)) - 1|^2 dM(x, y) \\ &\quad + \textstyle\int_{x\neq y, (x,y)\in E} \Delta(\gamma^a(x), \gamma^b(y)) dM(x, y) \\ &= \mu(\{x : \gamma^a(x) \neq \gamma^b(x)\}) \\ &\quad + M(\{(x, y) \in E : x \neq y \ \&\ \gamma^a(x) = \gamma^b(y)\}). \end{aligned}$$

Now

$$\begin{aligned} &M(\{(x,y) \in E : x \neq y \ \&\ \gamma^a(x) = \gamma^b(y)\}) \\ &= M(\{(x,y) \in E : x \neq y \ \&\ (\gamma^{-1})^b\gamma^a(x) = y\}) \\ &= \mu(\{x : (\gamma^{-1})^b\gamma^a(x) \neq x\}) = \mu(\{x : \gamma^a(x) \neq \gamma^b(x)\}) \\ &= \delta_u(\gamma^a, \gamma^b), \end{aligned}$$

so

$$\|\gamma^{-1} \cdot \Delta - \Delta\|_2^2 = 2\delta_u(\gamma^a, \gamma^b).$$

Recall now that Γ admits a Kazhdan pair (Q, ϵ), i.e., a pair consisting of a finite subset $Q \subseteq \Gamma$ and $\epsilon > 0$ such that for any unitary representation $\pi : \Gamma \to U(H)$, if there is $v \in H$ which is (Q, ϵ)-invariant, i.e., $\|\pi(\gamma)(v) - v\| < \epsilon\|v\|, \forall \gamma \in Q$, then there is $v \neq 0$ which is Γ-invariant. Recall also (see, e.g., Bekka-de la Harpe-Valette [BdlHV]) that if (Q, ϵ) is a Kazhdan pair for Γ and $\delta > 0$, then for any unitary representation $\pi : \Gamma \to U(H)$, if $v \in H$ is $(Q, \delta\epsilon)$-invariant, then there is Γ-invariant v_1 such that $\|v - v_1\| \leq \delta\|v\|$.

So given $\delta > 0$, choose $\eta > 0$ small enough such that if $\delta_{\Gamma,u}(a, b) < \eta$, then $2\delta_u((\gamma^{-1})^a, (\gamma^{-1})^b) < \delta^2\epsilon^2, \forall \gamma \in Q$. Then, applying the above to the representation of Γ on $L^2(E, M)$ and the vector Δ, we can find $f \in L^2(E, M)$ with $\|\Delta - f\|_2 \leq \delta$ which is Γ-invariant, i.e.,

$$f(\gamma^a(x), \gamma^b(y)) = f(x, y), \forall \gamma \in \Gamma.$$

Following Hjorth [Hj4] put

$$R(x, y) \Leftrightarrow xEy \ \&\ |f(x, y) - 1| < 1/2.$$

Then clearly R is $a \times b$-invariant, i.e.,

$$R(x, y) \Leftrightarrow R(\gamma^a(x), \gamma^b(y)).$$

Let

$$C = \{x : \text{ There is unique } y \text{ with } R(x, y)\},$$

and for $x \in C$, put

$$T(x) = \text{ the unique } y \text{ with } R(x, y).$$

Let also

$$D = T(C).$$

Clearly C is a-invariant, D is b-invariant and $T(\gamma^a(x)) = \gamma^b(T(x))$, for $x \in C$. Let now

$$B = \{y \in D : \text{ There is unique } x \in C \text{ with } R(x, y)\},$$

Then T^{-1} is well-defined on B and put

$$A = T^{-1}(B) \subseteq C.$$

Clearly A is a-invariant, B is b-invariant and if $\varphi = T|A$, then $\varphi \in [[E]]$ and $\varphi(a|A)\varphi^{-1} = b|B$. It remains to prove that $\mu(A)\ (= \mu(B)) \geq 1 - 16\delta^2$.

Claim. $\mu(D) \geq 1 - 12\delta^2$.

Proof of the claim. Let $D_1 = \{y : \forall x(x, y) \notin R\}, D_2 = \{x : \exists z \neq xR(x, z)\}, D_3 = \{y : \exists x \neq yR(x, y)\}$. We claim that

$$X \setminus D \subseteq D_1 \cup D_2 \cup D_3.$$

Indeed, let $y \notin D$ and $y \notin D_1$. Then there is x with $R(x, y)$, thus $x \notin C$, so there is $z \neq y$ with $R(x, z)$. If $x = y$, then clearly $y \in D_2$, while if $x \neq y$, then $y \in D_3$.

Recall now that for any Borel $h \geq 0, \int h dM = \int (\sum_{x \in [y]_E} h(x, y)) d\mu(y)$. Thus

$$\begin{aligned} \delta^2 \geq \|f - \Delta\|_2^2 &= \int \sum_{x \in [y]_E} |f(x, y) - \Delta(x, y)|^2 d\mu(y) \\ &\geq \int_{D_1} \tfrac{1}{4} d\mu(y) = \tfrac{\mu(D_1)}{4}, \end{aligned}$$

so $\mu(D_1) \leq 4\delta^2$. Also

$$\delta^2 \geq \|f - \Delta\|_2^2 \geq \int_{D_3} \frac{1}{4} d\mu(y) = \frac{\mu(D_3)}{4},$$

so $\mu(D_3) \leq 4\delta^2$.

Since also for Borel $h \geq 0, \int h dM = \int (\sum_{y \in [x]_E} h(x, y)) d\mu(x)$, and $D_2 = \{x : \exists y \neq xR(x, y)\}$, we have

$$\begin{aligned} \delta^2 \geq \|f - \Delta\|_2^2 &= \int \sum_{y \in [x]_E} |f(x, y) - \Delta(x, y)|^2 d\mu(x) \\ &\geq \int_{D_2} \tfrac{1}{4} d\mu = \tfrac{\mu(D_2)}{4}, \end{aligned}$$

thus $\mu(D_2) \leq 4\delta^2$, and finally $\mu(X \setminus D) \leq 12\delta^2$ and $\mu(D) \geq 1 - 12\delta^2$.

Let now

$$F = \{y : \exists x_1 \neq x_2(x_1, x_2 \in C \ \& \ R(x_1, y) \ \& \ R(x_2, y))\}.$$

Again F is b-invariant and as before

$$\delta^2 \geq \|f - \Delta\|_2^2 \geq \int_F \frac{1}{4} d\mu(y) = \frac{\mu(F)}{4},$$

so $\mu(F) \leq 4\delta^2$. Clearly $B = D \setminus F$ and thus $\mu(B) \geq 1 - 12\delta^2 - 4\delta^2 = 1 - 16\delta^2$.

For the final assertion of the lemma, note that $\{x \in A : \varphi(x) \neq x\} \subseteq D_2$. $\square$

We now have the following consequence.

Theorem 14.2. *Let* Γ *be a countable group with property* (T). *Then*
(i) $\mathrm{ERG}(\Gamma, X, \mu)$ *is clopen in* $(A(\Gamma, X, \mu), u)$,
(ii) Every conjugacy class in $\mathrm{ERG}(\Gamma, X, \mu)$ *is clopen in* $(A(\Gamma, X, \mu), u)$,
(iii) $\mathrm{WMIX}(\Gamma, X, \mu)$ *is clopen in* $(A(\Gamma, X, \mu), u)$.

Proof. Let $a \in \mathrm{ERG}(\Gamma, X, \mu)$ and let η be chosen as in Lemma 14.1 for $\delta < 1/4$. Let then $\delta_{\Gamma,u}(a, b) < \eta$. Then for φ, A, B as in that lemma, $\mu(A) > 0$, so by the ergodicity of $a, \mu(A) = \mu(B) = 1$ and so $\varphi \in \mathrm{Aut}(X, \mu)$ and $\varphi a \varphi^{-1} = b$, thus b is conjugate to a and b is ergodic. It follows that $\mathrm{ERG}(\Gamma, X, \mu)$ is open in u and so is every conjugacy class in $\mathrm{ERG}(\Gamma, X, \mu)$.

Now $\mathrm{ERG}(\Gamma, X, \mu)$ is also closed in u, by 12.2 i), and thus it is clopen in u and therefore the same is true for each conjugacy class contained in it.

That $\mathrm{WMIX}(\Gamma, X, \mu)$ is clopen in $A(\Gamma, X, \mu)$ follows from the fact that $a \in \mathrm{WMIX}(\Gamma, X, \mu)$ iff $a \times a \in \mathrm{ERG}(\Gamma, X \times X, \mu \times \mu)$ and the map $a \mapsto a \times a$ is uniformly continuous. □

A similar argument shows that for groups Γ with property (T), the set $\mathrm{FED}(\Gamma, X, \mu)$ is open and each conjugacy class contained in it is open in $(A(\Gamma, X, \mu), u)$.

It is clear from 14.2, and the separability of each full group in the uniform topology, that we now have the following.

Theorem 14.3 (Hjorth). *Let* Γ *have property* (T). *Then every orbit equivalence class in* $\mathrm{ERG}(\Gamma, X, \mu)$ *contains only countably many conjugacy classes.*

Hjorth derives from this a result about the complexity of classification of ergodic actions of Γ up to orbit equivalence.

Theorem 14.4 (Hjorth). *Let* Γ *be a countable infinite group with property* (T). *Then orbit equivalence of measure preserving, free, ergodic actions on* Γ *cannot be classified by countable structures.*

Proof. Using the notation and the argument in the proof of 13.7, and noting that, as Γ has property (T), Γ is not abelian-by-finite, we conclude, if the theorem fails, that there is $a \in \mathrm{FRERG}(\Gamma, X, \mu)$ such that we have $(a_0 \times a_\pi)\mathrm{OE}a$ for comeager many $\pi \in G_{\Gamma,H}$. Then, by 14.3, there is $\{a_n\} \subseteq \mathrm{FRERG}(\Gamma, X, \mu)$ such that on a comeager set of $\pi \in G_{\Gamma,H}, a_0 \times a_\pi \cong a_n$ for some n. So there is $b \in \mathrm{FRERG}(\Gamma, X, \mu)$ such that for a non-meager set of $\pi \in G_{\Gamma,H}, a_0 \times a_\pi \cong b$, from which we have a contradiction as in 13.7. □

Remark. I do not know what countable groups Γ have the property that $\mathrm{ERG}(\Gamma, X, \mu)$ is closed in u and similarly what groups Γ have the property that $\mathrm{ERG}(\Gamma, X, \mu)$ is open in u. It may very well be that the last property characterizes the property (T) groups. However, it is easy to see that if N_1, resp., N_2, denote the classes of groups Γ for which $\mathrm{ERG}(\Gamma, X, \mu)$ is not uniformly closed, resp., not uniformly open, then $\mathbb{Z} \in N_i$, and if $\Gamma \in N_i$ and Γ is a factor of Δ, then $\Delta \in N_i$, for $i = 1, 2$. For example, for the free group F_n $(n \geq 1)$, $\mathrm{ERG}(F_n, X, \mu)$ is neither closed nor open in u. In fact, it is not hard to see that $\mathrm{ERG}(F_n, X, \mu)$ is not even F_σ in u (it is of course G_δ in u). Indeed, fixing free generators $\gamma_1, \dots, \gamma_n$ for F_n, consider the closed in u set C of all actions a such that γ_i^a is aperiodic for all $i \leq n$. Then clearly the ergodic actions are uniformly dense in C by 2.4. But also the non-ergodic actions are uniformly dense in C (which, be the Baire Category Theorem implies that the ergodic actions cannot be F_σ in u). There are several ways to see this but perhaps the easiest, pointed out by Tsankov, is to use the proof of 2.8 and notice that the set A is invariant under the induced transformation T_A.

Remark. There is a version of 14.1 valid for an *arbitrary* countable group Γ. Consider the space $A(\Gamma, X, \mu)$ and define the metric

$$\delta_{\Gamma,\infty}(a,b) = \sup_{\gamma \in \Gamma} \delta_u(\gamma^a, \gamma^b).$$

Clearly $\delta_{\Gamma,\infty} \geq \delta_{\Gamma,u}$ and the topology induced by $\delta_{\Gamma,\infty}$ contains the uniform topology. It is also easy to check that $\delta_{\Gamma,\infty}$ is a complete metric on $A(\Gamma, X, \mu)$.

Given $a, b \in A(\Gamma, X, \mu)$ such that $\delta_{\Gamma,\infty}(a,b) < \delta^2/2$ and going through the proof of 14.1 we see that $\|\gamma \cdot \Delta - \Delta\|_2 < \delta, \forall \gamma \in \Gamma$. Thus $\Gamma \cdot \Delta$ is a Γ-invariant set, contained in the ball of radius δ around Δ. Let f be the unique element of least norm in the closed convex hull of $\Gamma \cdot \Delta$. Then f is Γ-invariant and $\|\Delta - f\| \leq \delta$. The rest of the proof of 14.1 can be repeated verbatim. Thus we have that for any countable group Γ and $a, b \in A(\Gamma, X, \mu)$, if $\delta_{\Gamma,\infty}(a,b) < \delta^2/2$ then there is $\varphi : A \to B, \varphi \in [[E_a \vee E_b]]$ such that A is a-invariant, B is b-invariant, $\varphi(a|A)\varphi^{-1} = b|B$ and $\mu(A) > 1 - 16\delta^2, \mu(\{x \in A : \varphi(x) \neq x\}) \leq 4\delta^2$.

It follows, as in 14.2, that for any countable group Γ, $\mathrm{ERG}(\Gamma, X, \mu)$ is clopen in the $\delta_{\Gamma,\infty}$-topology and so is every conjugacy class in $\mathrm{ERG}(\Gamma, X, \mu)$. Similarly $\mathrm{WMIX}(\Gamma, X, \mu)$ is clopen in this topology.

Finally, we note that if Γ has property (T), then the uniform topology coincides with the $\delta_{\Gamma,\infty}$-topology. For this it is enough to show that for any $\delta > 0$, there is $\eta > 0$ such that if $\delta_{\Gamma,u}(a,b) < \eta$, then $\delta_{\Gamma,\infty}(a,b) \leq 56\delta^2$. Fix $\delta > 0$ and let η be given by 14.1. If $\delta_{\Gamma,u}(a,b) < \eta$ find $\varphi : A \to B$ as in Lemma 14.1 and extend φ to $T \in \mathrm{Aut}(X, \mu)$, so that $T(X \setminus A) = X \setminus B$. Now for each $\gamma \in \Gamma, T\gamma^a T^{-1}(x) = \gamma^b(x)$, for each $x \in B$, thus $\delta_u(\gamma^b, T\gamma^a T^{-1}) \leq \mu(X \setminus B) < 16\delta^2$ and $\delta_u(T, 1) \leq \mu(\{x \in A : \varphi(x) \neq x\} \cup (X \setminus A)) < 20\delta^2$. Thus $\delta_u(\gamma^b, \gamma^a) \leq \delta_u(\gamma^b, T\gamma^a T^{-1}) + \delta_u(T\gamma^a T^{-1}, \gamma^a) \leq 16\delta^2 + 2\delta_u(T, 1) \leq 56\delta^2$, so $\delta_{\Gamma,\infty}(a,b) \leq 56\delta^2$.

(B) Lemma 14.1 also has implications concerning the outer automorphism group of equivalence relations induced by actions of property (T) groups.

Theorem 14.5 (Gefter-Golodets [GG]). *Suppose the countable group* Γ *has property* (T) *and let* $a \in \mathrm{ERG}(\Gamma, X, \mu)$. *Denote by* $C_a \subseteq N[E_a]$ *the* centralizer *of* a, *i.e., the set of all* $T \in \mathrm{Aut}(X, \mu)$ *that preserve the action:* $TaT^{-1} = a$. *Then* $C_a[E_a] = \{TS : T \in C_a, S \in [E_a]\}$ *is clopen in* $N[E_a]$. *In particular, if* $C_a = \{1\}$, *then* $\mathrm{Out}(E_a)$ *is countable.*

Proof. Put $E = E_a, C = C_a$. For $T \in N[E]$, let

$$\rho(T) = \sum 2^{-n} \delta_u(\gamma_n^a, T\gamma_n^a T^{-1}),$$

where $\Gamma = \{\gamma_n\}$. Then it is enough to show that there is some $\eta > 0$ such that for $T \in N[E]$,

$$\rho(T) < \eta \Rightarrow T \in C[E].$$

Put $b = TaT^{-1}$. Then $E_b \subseteq E$. Moreover

$$\rho(T) = \delta_{\Gamma,u}(a, b).$$

So if η is as in 14.1 for $\delta < \frac{1}{4}$ and $\rho(T) < \eta$, then (as a, b are ergodic and $E_a \vee E_b = E$), there is $S \in [E]$ with $SaS^{-1} = b = TaT^{-1}$, and thus we have $(S^{-1}T)a(S^{-1}T)^{-1} = a$, i.e., $S^{-1}T \in C$, so $T \in SC \subseteq [E]C = C[E]$. □

Note also the following result.

Theorem 14.6. *Let Γ be a countable group and let $a \in \mathrm{FRERG}(\Gamma, X, \mu)$. Then $C_a \cap [E_a] = \{1\}$ if one of the two conditions below hold:*

(i) (**Gefter [Ge]**) *Γ is* ICC.

(ii) Γ is centerless and every infinite subgroup of Γ acts ergodically (e.g., the action is mixing).

Proof. Suppose $T \in [E_a]$ is in C_a. Let $T = \bigcup_i \gamma_i | A_i$, where $A = \bigcup_i A_i$ is a partition of X into Borel sets of positive measure. By ergodicity A_i meets every E_a-class. We will show that each $\gamma_i = 1$.

For $xE_a y$, let

$$\alpha(x, y) \cdot x = y$$

be the corresponding cocycle $\alpha : E_a \to \Gamma$. For each E_a-class C, let

$$\Gamma_i^C = \{\alpha(x, y) : x, y \in C \cap A_i\} \subseteq \Gamma.$$

Then by ergodicity $\Gamma_i^C = \Gamma_i$ is constant for (almost) all C.

Claim. *If $\delta \in \Gamma_i$, then $\gamma_i \delta = \delta \gamma_i$.*

Proof of the claim. Indeed, if $\delta = \alpha(x, y)$, where $x, y \in A_i, xE_a y$, we have $\delta \cdot x = y$ and

$$\gamma_i \delta \cdot x = \gamma_i \cdot y = T(y) = T(\delta \cdot x) = \delta \cdot T(x) = \delta \gamma_i \cdot x.$$

Since the action is free, $\gamma_i \delta = \delta \gamma_i$.

Thus γ_i commutes with every element of $\langle \Gamma_i \rangle$. Consider first case (ii). Since A_i meets every E_a-class infinitely often, Γ_i is infinite and so is $\langle \Gamma_i \rangle$, thus $\langle \Gamma_i \rangle$ acts ergodically. Now $\langle \Gamma_i \rangle \cdot A_i$ is conull and thus contains a conull E_a-invariant set, thus for (almost) all C, C is a $\langle \Gamma_i \rangle$-orbit, thus (again by the freeness of the action) $\langle \Gamma_i \rangle = \Gamma$. So $\gamma_i = 1$.

In case (i), notice that for the conjugacy action of Γ on itself the stabilizer of γ_i contains $\langle \Gamma_i \rangle$. If $\gamma_i \neq 1$, the conjugacy class of γ_i is infinite, so $\langle \Gamma_i \rangle$ has infinite index in Γ, which implies that there are infinitely many $\{\delta_n\} \subseteq \Gamma$ with $\{\delta_n \langle \Gamma_i \rangle \cdot A_i\}$ pairwise disjoint, a contradiction.

(There is actually a simpler proof in case (i); see Golodets [Gol] or Furman [Fu1]: Suppose $T \in C_a \cap [E_a]$. There is Borel $f(x) = \gamma_x \in \Gamma$ with $T(x) = \gamma_x \cdot x$. Then $\gamma_{\delta \cdot x} = \delta \gamma_x \delta^{-1}$. Thus $f_* \mu$ is a probability measure on Γ which in conjugacy-invariant, so, since Γ is ICC, concentrates on 1, i.e., $f(x) = 1$ and so $T = 1$.) □

(C) Finally, one can use these ideas to give a proof of the result of Ozawa [O] that no full group $[F]$ contains (algebraically) every countable group. Assume such F existed, towards a contradiction, and without loss of generality take F to be ergodic (by enlarging it if necessary). Let us say that a non-trivial countable group $\Gamma \leq [F]$ is *nice* if $\{x : \gamma(x) = x, \forall \gamma \in \Gamma\}$ has measure $< \frac{1}{2}$. Notice that if $\Gamma \leq [F]$, then there is an isomorphic copy Γ' of $\Gamma, \Gamma' \leq [F]$, such that Γ' is nice. Indeed, if Γ is not nice itself, let $C = \{x : \gamma(x) = x, \forall \gamma \in \Gamma\}$. Then $1 > \mu(C) \geq \frac{1}{2}$, so if $D = X \setminus C,\ 0 < \mu(D) \leq \frac{1}{2}$. Note that, by the ergodicity of F, both C, D are complete sections of F. Let $C_1, \dots, C_n$ be pairwise disjoint subsets of C such that $\mu(C_i) = \mu(D)$ and $\mu(C \setminus \bigcup_{i=1}^n C_i) < \frac{1}{2}$. Let $\varphi_i : D \to C_i$ be a Borel 1-1 correspondence with $\varphi_i(x) F x, \forall x \in D$. Notice that C, D and each C_i are of course Γ-invariant. For each $\gamma \in \Gamma$, let $\gamma' \in [F]$ be defined as follows: $\gamma'(x) = \gamma(x)$, if $x \in D; \gamma'(x) = \gamma(x) = x$, if $x \in C \setminus \bigcup_{i=1}^n C_i; \gamma'(x) = \varphi_i(\gamma(\varphi_i^{-1}(x))$, if $x \in C_i$. Clearly $\gamma \mapsto \gamma'$ is an isomorphism of Γ with $\Gamma' \leq [F]$ and $\{x : \gamma'(x) = x, \forall \gamma \in \Gamma\} = C \setminus \bigcup_{i=1}^n C_i$, so Γ' is nice.

As in Ozawa [O], we fix a countable property (T) group Γ which has uncountably many pairwise non-isomorphic factors $\{\Gamma_i\}_{i \in I}$ with each Γ_i simple. Fix epimorphisms $\pi_i : \Gamma \to \Gamma_i$. Then the kernels $N_i = \ker(\pi_i), i \in I$, are all distinct. Assume, towards a contradiction, that each Γ_i can be (algebraically) embedded into $[F]$. By the preceding remarks, we can assume that the image of each embedding is nice, i.e., there is an isomorphism $\varphi_i : \Gamma_i \to [F]$ with $\varphi_i(\Gamma_i)$ nice. Consider then the action a_i of Γ on (X, μ) given by $\gamma^{a_i} = \varphi_i(\pi_i(\gamma))$. Thus for all i, $\mu(\{x : \gamma^{a_i}(x) = x, \forall \gamma \in \Gamma\}) < \frac{1}{2}$. Clearly $\{a \in A(\Gamma, X, \mu) : \gamma^a \in [F], \forall \gamma \in \Gamma\}$ is separable in the uniform topology, so there are $i \neq j$ with $\delta_{\Gamma,u}(a_i, a_j) < \eta$, where η is chosen small enough so that, by Lemma 14.1, there is $\varphi : A \to B, \varphi \in [[F]]$ with A a_i-invariant, B a_j-invariant, $\varphi(a_i|A)\varphi^{-1} = (a_j|B)$, and $\mu(A) > \frac{1}{2}$. Since $N_i \neq N_j$, suppose, without loss of generality, that $N_j \not\subseteq N_i$. Then $\pi_i(N_j)$ is a non-trivial normal subgroup of Γ_i, so $\pi_i(N_j) = \Gamma_i$. But if $\gamma \in N_j, \gamma^{a_j}|B = \text{id}$, so $\gamma^{a_i}|A = \text{id}$, thus $A \subseteq \{x : \gamma^{a_i}(x) = x, \forall \gamma \in \Gamma\}$, which is a contradiction, as $\mu(A) > \frac{1}{2}$.

15. Connectedness in the space of actions

(A) Recall from 2.8 that the group $\text{Aut}(X, \mu)$, which we can identify with the space of $\mathbb{Z}$-actions, $A(\mathbb{Z}, X, \mu)$, is contractible in the uniform topology and the same argument shows that each open ball in δ_u around the identity is also contractible, so that in particular $(\text{Aut}(X, \mu), u)$ is locally path connected. By 2.9, $(\text{Aut}(X, \mu), w)$ is homeomorphic to ℓ^2, so again it is locally path connected. We next prove another local connectedness fact about $\text{Aut}(X, \mu)$ which can serve as a prototype for similar results for other $A(\Gamma, X, \mu)$ that we will discuss later on.

We start with a lemma that can be proved by using the argument for turbulence in 5.1. Below for $T \in \text{Aut}(X, \mu), \epsilon > 0$, we put

$$V_{T,\epsilon} = \{S \in \text{Aut}(X, \mu) : \delta_w(S, T) < \epsilon\},$$

where δ_w is the metric giving the weak topology on $\mathrm{Aut}(X,\mu)$, as in Section 1, **(B)**.

Lemma 15.1. *Let* $T \in \mathrm{Aut}(X,\mu), T \in [E]$, *with* E *hyperfinite,* $g \in [E], \delta, \epsilon > 0$ *and* $gTg^{-1} \in V_{T,\epsilon}$. *Then there is a continuous path in* $([E],u)$, $\lambda \mapsto g_\lambda, \lambda \in [0,1]$, *with* $g_0 = 1$ *and* $g_\lambda T g_\lambda^{-1} \in V_{T,\epsilon}$, *for which we have* $\delta'_u(gTg^{-1}, g_1Tg_1^{-1}) < \delta$. *Similarly if* $V_{T,\epsilon}$ *is replaced by the open ball of radius* ϵ *around* T *in the metric* δ'_u.

Using this one can prove the following fact, due to Rosendal and the author.

Lemma 15.2. *Let* $T \in \mathrm{APER}, T \in [E]$, *with* E *ergodic, hyperfinite, and let* $S \in V_{T,\epsilon}$. *Then there is a continuous map* $\lambda \mapsto h_\lambda, \lambda \in [0,1)$, *in* $([E],u)$ *with* $h_0 = 1, h_\lambda T h_\lambda^{-1} \in V_{T,\epsilon}$ *and* $h_\lambda T h_\lambda^{-1} \to S$ *weakly as* $\lambda \to 1$. *In particular, there is a continuous path in* $(V_{T,\epsilon}, w)$ *connecting* T *to* S.

Proof. Let $V_\epsilon = V_{T,\epsilon}$. Fix a sequence $\epsilon_n \downarrow 0$ such that $V_{S,2\epsilon_n} \subseteq V_\epsilon, \forall n$. By the aperiodicity of T and the ergodicity of E, $\{gTg^{-1} : g \in [E]\}$ is weakly dense in $\mathrm{Aut}(X,\mu)$, so we can find $g \in [E]$ such that gTg^{-1} is as close to S as we want, in the weak topology, so by 15.1 we can find a continuous path $h_{1,\lambda}$ in $([E],u)$ such that $h_{1,0} = 1, h_{1,\lambda}Th_{1,\lambda}^{-1} \in V_\epsilon, T_1 = h_{1,1}Th_{1,1}^{-1} \in V_{S,\epsilon_1}$, and so $S \in V_{T_1,\epsilon_1}$, thus $V_{T_1,\epsilon_1} \subseteq V_{S,2\epsilon_1} \subseteq V_\epsilon$. By the same argument, we can find a continuous path $h_{2,\lambda}$ in $([E],u)$ such that $h_{2,0} = 1, h_{2,\lambda}T_1h_{2,\lambda}^{-1} \in V_{T_1,\epsilon_1}, T_2 = h_{2,1}T_1h_{2,1}^{-1} \in V_{S,\epsilon_2}$, and so $S \in V_{T_2,\epsilon_2}$, thus $V_{T_2,\epsilon_2} \subseteq V_{S,2\epsilon_2} \subseteq V_\epsilon, \dots$ Fix finally $\alpha_1 < \alpha_2 < \cdots \to 1$. Then define h_λ as follows:

$$h_\lambda = h_{1,\frac{\lambda}{a_1}}, \lambda \in [0,a_1],$$
$$h_\lambda = h_{2,\frac{(\lambda-a_1)}{(a_2-a_1)}} h_{1,1}, \lambda \in [a_1,a_2],$$
$$h_\lambda = h_{3,\frac{(\lambda-a_3)}{(a_3-a_2)}} h_{2,1}h_{1,1}, \lambda \in [a_2,a_3], \dots$$

It is easy to see that this works. □

Theorem 15.3. *Let* $C \subseteq \mathrm{Aut}(X,\mu)$ *contain the conjugacy class of an aperiodic* $T \in \mathrm{Aut}(X,\mu)$. *Then* (C,w) *is path connected and locally path connected.*

Proof. From 15.2, by taking $\epsilon > 1$, and using also 5.4, we see that C is path connected. Also by considering the family of all $V_{T,\epsilon} \cap C$ with $T \in \mathrm{APER} \cap C$, we see that it is locally path connected. □

A similar argument, using now the proof of 5.2, shows the following.

Theorem 15.4. *Let* E *be an ergodic equivalence relation which is not* E_0-*ergodic. Let* $C \subseteq \mathrm{APER} \cap [E]$ *contain the conjugacy class in* $[E]$ *of an aperiodic* $T \in [E]$. *Then* (C,u) *is path connected and locally path connected. Moreover, a continuous path from* T *to any* $S \in C$ *can be found that has* *the form* $T_\lambda = g_\lambda T g_\lambda^{-1}, \lambda \in [0,1), T_1 = S$, *where* $\lambda \mapsto g_\lambda, \lambda \in [0,1)$, *is continuous in* $([E], u)$.

Corollary 15.5. *Both* (APER, u) *and* (ERG, u) *are path connected.*

Proof. Let $S, T \in \mathrm{ERG}$. By Dye's Theorem 3.13, there is $U \in \mathrm{Aut}(X,\mu)$ such that $U[E_S]U^{-1} = [E_T]$ and thus $USU^{-1} \in [E_T]$. Since $USU^{-1}, T \in [E_T]$, it follows from 15.4 that there is a continuous path in $(\mathrm{ERG} \cap [E_T], u)$ from USU^{-1} to T. By 2.8, there is a continuous path $U_\lambda, 0 \leq \lambda \leq 1$, in $(\mathrm{Aut}(X,\mu), u)$ from 1 to U. Clearly $U_\lambda S U_\lambda^{-1}$ is a continuous path in (ERG, u) from S to USU^{-1}, so there is a continuous path in (ERG, u) from S to T.

The argument for APER is similar (using the fact, see 5.4, that for any $S, T \in \mathrm{APER}$, there are $S_0, T_0 \in \mathrm{ERG}$ with $E_S \subseteq E_{S_0}, E \subseteq E_{T_0}$). As pointed out by Tsankov, it can be also proved directly as in 2.8. □

(B) We now consider a countable group Γ and the space $A(\Gamma, X, \mu)$. First let us note the following fact.

Proposition 15.6. *For any countable group* Γ, *the conjugacy class of any* $a \in A(\Gamma, X, \mu)$ *is path connected in* $(A(\Gamma, X, \mu), u)$ (*and thus also in the space* $(A(\Gamma, X, \mu), w)$).

Proof. Let $a \in A(\Gamma, X, \mu)$ and $T \in \mathrm{Aut}(X,\mu)$. By 2.8, there is a continuous path $t \mapsto T_t$ in $(\mathrm{Aut}(X,\mu), u)$ with $T_0 = 1, T_1 = T$. Let $a_t = T_t a T_t^{-1}$. Then $t \mapsto a_t$ is a continuous path in $(A(\Gamma, X, \mu), u)$ with $a_0 = a, a_1 = TaT^{-1}$. □

Corollary 15.7 (a weak version of 15.11). *For any countable group* Γ, *the space* $(A(\Gamma, X, \mu), w)$ *is connected.*

Proof. From 10.7 and 15.6. □

This fact can be reformulated as follows. For any topological group G, a *variety* on G is a closed subset on G^N, where $N = 1, 2, \dots, \mathbb{N}$, of the form

$$\{(g_1, g_2, \dots) \in G^N : \forall i \in I (w_i^{g_1, g_2 \cdots} = 1)\},$$

where $\{w_i\}_{i \in I}$ is a family of words in the free group F_N. Recalling the remark in Section 10, **(A)**, we can rephrase 15.7 as follows: *Every variety in* $(\mathrm{Aut}(X,\mu), w)$ *is connected.*

The next question is whether $(A(\Gamma, X, \mu), w)$ is path connected. The answer turns out to be positive. The main fact is that it holds for groups that do not have property (T). In fact, for such groups we have a complete analog of 15.3.

Theorem 15.8 (Kechris). *Let* Γ *be a countable group that does not satisfy property* (T). *Let* $C \subseteq A(\Gamma, X, \mu)$ *contain the conjugacy class of an* $a \in A(\Gamma, X, \mu)$, *which is ergodic, not* E_0*-ergodic and has a weakly dense conjugacy class. Then* (C, w) *is path connected and locally path connected. In particular,* $(A(\Gamma, X, \mu), w)$ *has these properties.*

Proof. For $\vec{\gamma} = \{\gamma_1, \ldots, \gamma_\ell\} \subseteq \Gamma, \vec{B} = \{B_1, \ldots, B_k\} \subseteq \mathrm{MALG}_\mu, \epsilon > 0, a \in A(\Gamma, X, \mu)$, let

$$U_{a,\epsilon,\vec{\gamma},\vec{B}} = \{b \in A(\Gamma, X, \mu) : \forall j \leq \ell, \forall i \leq k(\mu(\gamma_j^a(B_i)\Delta\gamma_j^b(B_i)) < \epsilon)\}.$$

Let also $\delta'_{\Gamma,u}$ on $A(\Gamma, X, \mu)$ be defined by

$$\delta'_{\Gamma,u}(a,b) = \sum 2^{-n}\delta'_u(\gamma_n^a, \gamma_n^b),$$

for some fixed enumeration $\Gamma = \{\gamma_n\}$, so that $\delta'_{\Gamma,u}$ is a complete compatible metric in u.

First we have an analog of 15.1, whose proof is as in 13.3.

Lemma 15.9. *Let $a \in A(\Gamma, X, \mu)$ be ergodic but not E_0-ergodic, $T \in [E_a], \delta > 0$ and $TaT^{-1} \in U_{a,\epsilon,\vec{\gamma},\vec{A}}$. Then there is a continuous path in $([E_a], u), \lambda \mapsto T_\lambda, \lambda \in [0,1]$, with $T_0 = 1$, $T_\lambda a T_\lambda^{-1} \in U_{a,\epsilon,\vec{\gamma},\vec{A}}$ and*

$$\delta'_{\Gamma,u}(T_1 a T_1^{-1}, TaT^{-1}) < \delta.$$

From this, as in 15.2, we can deduce the following lemma.

Lemma 15.10. *Let $a \in \mathrm{ERG}(\Gamma, X, \mu) \setminus \mathrm{E_0RG}(\Gamma, X, \mu)$ have weakly dense conjugacy class in $A(\Gamma, X, \mu)$. Let $b \in U_{a,\epsilon,\vec{\gamma},\vec{A}}$. Then there is continuous $\lambda \mapsto T_\lambda, \lambda \in [0,1)$, in $(\mathrm{Aut}(X,\mu), u)$ with $T_0 = 1, T_\lambda a T_\lambda^{-1} \in U_{a,\epsilon,\vec{\gamma},\vec{A}}$ and $T_\lambda a T_\lambda^{-1} \to b$ weakly as $\lambda \to 1$. In particular, there is a continuous path in $(U_{a,\epsilon,\vec{\gamma},\vec{A}}, w)$ connecting a to b.*

The theorem is then immediate from 15.10. The last assertion also follows using 10.7, 12.4 ii). □

Theorem 15.11 (Kechris). *Let Γ be a countable group. Then the space $(A(\Gamma, X, \mu), w)$ is path connected and locally path connected.*

Proof. Let $\Delta = \Gamma * \mathbb{Z}$, so that Δ does not have property (T). Then by Section 10, **(G)**, $A(\Delta, X, \mu)$ is homeomorphic to $(A(\Gamma, X, \mu) \times A(\mathbb{Z}, X, \mu)$, and since, by 15.8, $A(\Delta, X, \mu)$ is path connected and locally path connected, so is $A(\Gamma, X, \mu)$. □

Again this can be reformulated as: *Every variety in* $(\mathrm{Aut}(X, \mu), w)$ *is path connected and locally path connected.*

It follows from the proofs of 15.8 and 15.11 that for any Γ, the space $(\mathrm{FR}(\Gamma, X, \mu), w)$ is path connected. It is also clear from 15.8 that the space $(\mathrm{ERG}(\Gamma, X, \mu), w)$ is path connected, when Γ does not have property (T). I do not know if $(\mathrm{ERG}(\Gamma, X, \mu), w)$ is path connected for any Γ.

(C) We next consider connectedness properties in $(A(\Gamma, X, \mu), u)$, when Γ has property (T).

Theorem 15.12. *Let the countable group Γ have property* (T). *Then for any $a \in \mathrm{ERG}(\Gamma, X, \mu)$, the path component of a in $(A(\Gamma, X, \mu), u)$ is exactly its conjugacy class and thus is clopen in $(A(\Gamma, X, \mu), u)$.*

Proof. By 15.6 the conjugacy class of any $a \in A(\Gamma, X, \mu)$ is contained in its uniform path component. By 14.2 the conjugacy class of $a \in \mathrm{ERG}(\Gamma, X, \mu)$ is clopen in the uniform topology, so it is exactly equal to its uniform path component. □

It thus follows that if an infinite Γ has property (T), $(\mathrm{ERG}(\Gamma, X, \mu), u)$ and also $(\mathrm{FRERG}(\Gamma, X, \mu), u)$ is far from connected. On the other hand, we have seen in 15.5 that $(\mathrm{FRERG}(\mathbb{Z}, X, \mu), u) = (\mathrm{ERG}(\mathbb{Z}, X, \mu), u) = (\mathrm{ERG}, u)$ is path connected and in fact the same argument works in the following more general situation.

Theorem 15.13. *Let Γ be a countable amenable group. Then the set $\mathrm{FRERG}(\Gamma, X, \mu)$ is path connected in the uniform topology.*

Proof. As in the proof of 15.5 and using the fact that if $a \in A(\Gamma, X, \mu)$, then the equivalence relation E_a is hyperfinite (Ornstein-Weiss [OW]), it is enough to show that if $a, b \in \mathrm{FRERG}(\Gamma, X, \mu)$ and $E_a \subseteq E_b$, then there is a continuous path in the space $(\mathrm{FRERG}(\Gamma, X, \mu), u)$ from a to b. Now Foreman-Weiss [FW], proof of Claim 19, show that the set of conjugates of a by elements of $[E_b]$ contains in its uniform closure the action b. Then we can use the argument in the proof of 15.8. □

It would be interesting to characterize the countable groups Γ for which $(\mathrm{FRERG}(\Gamma, X, \mu), u)$ is path connected.

Let now

$$D(\Gamma, X, \mu) = \{a : \exists A(0 < \mu(A) \ \&\ A \text{ is } a\text{-invariant} \ \&\ a|A \text{ is ergodic})\}.$$

Thus D is the set of all actions whose ergodic decomposition has a “discrete” component, i.e., an ergodic component of positive measure. Thus $\mathrm{ERG}(\Gamma, X, \mu) \subseteq D(\Gamma, X, \mu)$. Let also

$$C(\Gamma, X, \mu) = A(\Gamma, X, \mu) \setminus D(\Gamma, X, \mu)$$

be the set of all actions with “continuous” ergodic decomposition.

Proposition 15.14. *Let Γ be a countable group. Then $C(\Gamma, X, \mu)$ is dense in $(A(\Gamma, X, \mu), w)$.*

Proof. Fix $b \in C(\Gamma, Y, \nu)$. Since, by 10.4, $a \in A(\Gamma, X, \mu)$ is in the weak closure of the set of actions isomorphic to $a \times b$, it is enough to show that the latter admits no invariant set of positive measure on which it is ergodic. Assume otherwise, and let $A \subseteq X \times Y$ be Borel, $a \times b$-invariant with $(\mu \times \nu)(A) > 0$ and $a \times b|A$ ergodic. Let $\sigma_A = \frac{(\mu\times\nu)|A}{(\mu\times\nu)(A)}, \pi : X \times Y \to Y$ the projection, and put $\rho = \pi_* \sigma_A$. Then ρ is a b-invariant, ergodic probability measure, so if Y_ρ is the ergodic component corresponding to $\rho, \nu(Y_\rho) = 0$ and $\rho(Y_\rho) = 1$. Then $(\mu \times \nu)(X \times Y_\rho) = 0$ and so $\rho(Y_\rho) = \sigma_A(\pi^{-1}(Y_\rho)) = \frac{\mu\times\nu((X\times Y_\rho)\cap A)}{(\mu\times\nu)(A)} = 0$, a contradiction. □

Proposition 15.15. *Let Γ be a countable group. Then $D(\Gamma, X, \mu)$ is dense in $(A(\Gamma, X, \mu), u)$.*

Proof. Let $a \in C(\Gamma, X, \mu)$. Let $\pi : X \to \mathcal{E}$ be the ergodic decomposition of a, i.e., $\mathcal{E}$ is the standard Borel space of a-invariant, ergodic measures, π is an a-invariant Borel surjection, and for each $e \in \mathcal{E}, e(\pi^{-1}(e)) = 1$ (and e is the unique a-invariant, ergodic measure on $\pi^{-1}(e)$). Let $\pi_*\mu = \nu$. Since $a \in C(\Gamma, X, \mu), \nu$ is non-atomic, thus we can find $A_t \in \mathrm{MALG}_\nu, 0 \le t \le 1$, with $A_0 = \emptyset, A_1 = \mathcal{E}, s \le t \Rightarrow A_s \subseteq A_t, \nu(A_t) = t$. Let $X_t = \pi^{-1}(A_t)$. Then X_t is a-invariant, $X_0 = \emptyset, X_1 = X, s \le t \Rightarrow X_s \subseteq X_t, \mu(X_t) = t$.

Fix now $1 > \epsilon > 0$ and let $Y_\epsilon = X \setminus X_\epsilon$. Fix a measure preserving, ergodic action a_ϵ of Γ on $(X_\epsilon, (\mu|X_\epsilon)/\epsilon)$ and let

$$b = a|Y_\epsilon \cup a_\epsilon.$$

Then $b \in D(\Gamma, X, \mu)$ and clearly $\delta_{\Gamma.u}(a, b) \le \epsilon$. □

A similar argument shows the following.

Proposition 15.16. *Let Γ be a countable group. Then $C(\Gamma, X, \mu)$ is path connected in the uniform topology and thus is contained in the path connected component of the trivial action $i_\Gamma \in A(\Gamma, X, \mu)$ in the uniform topology.*

Proof. In the notation of the preceding proof, let $a \in C(\Gamma, X, \mu)$ and put

$$a_t = a|X_t \cup i_\Gamma|Y_t, 0 \le t \le 1.$$

Then $t \mapsto a_t$ is a continuous path in $(C(\Gamma, X, \mu), u)$ which connects i_Γ to $a_1 = a$. □

Finally we have the following result.

Theorem 15.17 (Kechris). *If a countable group Γ has property (T), then $D(\Gamma, X, \mu)$ is open (and dense) in $(A(\Gamma, X, \mu), u)$ and $C(\Gamma, X, \mu)$ is exactly the path component of i_Γ in $(A(\Gamma, X, \mu), u)$.*

Proof. Let $a \in D(\Gamma, X, \mu)$ and fix an a-invariant set C with $\mu(C) > 0$ and $a|C$ ergodic. Next, by 14.1, there is $\eta > 0$ so that if $\delta_{\Gamma,u}(a, b) < \eta$, then for the $\varphi : A \to B$ given in the lemma, we have $\mu(A) > 1 - \mu(C)$, thus $\mu(A \cap C) > 0$. Since $A \cap C$ is a-invariant, it follows that $A \cap C = C$, so $C \subseteq A$ and since φ gives an isomorphism of $a|A$ with $b|B$, it follows that $\varphi(C)$ is b-invariant and $b|\varphi(C)$ is ergodic, so $b \in D(\Gamma, X, \mu)$ (and there is b-invariant set $D = \varphi(C)$ with $\mu(D) = \mu(C)$ and $b|D$ ergodic). This proves that $D(\Gamma, X, \mu)$ is open.

Finally we show that $C(\Gamma, X, \mu)$ is the uniform path connected component of i_Γ. Indeed, let $t \mapsto a_t$ be a uniformly continuous map from [0,1] to $A(\Gamma, X, \mu)$ with $a_0 = i_\Gamma$ and $a_1 = a \in D(\Gamma, X, \mu)$, towards a contradiction. Let C, η be as above and find $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$ such that $\delta_{\Gamma,u}(a_{t_i}, a_{t_{i+1}}) < \eta, \forall i < n$. By the preceding argument, a simple backwards induction shows that for every $i \le n, a_{t_i} \in D(\Gamma, X, \mu)$, so $i_\Gamma \in D(\Gamma, X, \mu)$, a contradiction. □

Remark. One can also give a more direct proof for the fact that $D(\Gamma, X, \mu)$ is open in $(A(\Gamma, X, \mu), u)$, when Γ has property (T). We start with the following lemma that we will also use in 15.20.

Lemma 15.18. *Suppose (Q, ϵ) is a Kazhdan pair for Γ. Let $a \in A(\Gamma, X, \mu)$, and fix $\delta > 0$. If a Borel set A satisfies $\mu(\gamma^a(A)\Delta A) < \delta\epsilon^2\mu(A), \forall\gamma \in Q$, then there is an a-invariant Borel set B with $\mu(A\Delta B) < 8\delta\mu(A)$.*

Proof. Consider the Koopman representation κ^a associated to a. Then we have $\|\kappa^a(\gamma)(\chi_A) - \chi_A\|^2 = \mu(\gamma^a(A)\Delta A), \|\chi_A\|^2 = \mu(A)$, so $\|\kappa^a(\gamma)(\chi_A) - \chi_A\| < \sqrt{\delta}\epsilon\|\chi_A\|, \forall\gamma \in Q$. Thus there is $f \in L^2(X, \mu)$ which is κ^a-invariant and $\|\chi_A - f\| \leq \sqrt{\delta}\|\chi_A\| = \sqrt{\delta\mu(A)}$. Let $B = \{x : |f(x) - 1| \leq \frac{1}{2}\}$. Then B is a-invariant and

$$x \in A \setminus B \Rightarrow |\chi_A(x) - f(x)| = |1 - f(x)| > \frac{1}{2},$$

so $\delta\mu(A) \geq \|\chi_A - f\|^2 = \int |\chi_A(x) - f(x)|^2 d\mu > \int_{A\setminus B} \frac{1}{4} d\mu = \frac{1}{4}\mu(A \setminus B)$. Also

$$x \in B \setminus A \Rightarrow |\chi_A(x) - f(x)| = |f(x)| \geq \frac{1}{2},$$

thus

$$\delta\mu(A) \geq \frac{1}{4}\mu(B \setminus A),$$

so $\frac{1}{4}\mu(A\Delta B) < 2\delta\mu(A)$ or $\mu(A\Delta B) < 8\delta\mu(A)$. □

To complete the proof fix $a \in D(\Gamma, X, \mu)$ and let $A \in \mathrm{MALG}_\mu$ be a-invariant with $\mu(A) > 0$ and $a|A$ ergodic. Let $\Gamma = \{\gamma_1, \gamma_2, \dots\}, Q = \{\gamma_1 \dots, , \gamma_N\}$ and recall that

$$\delta_{\Gamma,u}(a, b) = \sum_{n=1}^{\infty} 2^{-n}\delta_u(\gamma_n^a, \gamma_n^b) \geq 2^{-N} \sum_{\gamma\in Q} \delta_u(\gamma^a, \gamma^b).$$

We will see that if

$$\delta_{\Gamma,u}(a, b) < \frac{2^{-N}}{100}\epsilon^2\mu(A),$$

then $b \in D(\Gamma, X, \mu)$. Fix such an action b. Then $\mu(\gamma^a(A)\Delta\gamma^b(A)) = \mu(A\Delta\gamma^b(A)) \leq 2^N\delta_{\Gamma,u}(a, b) < \frac{\epsilon^2}{100}\mu(A), \forall\gamma \in Q$, so there is a b-invariant set $B \in \mathrm{MALG}_\mu$ with $\mu(A\Delta B) < \frac{8}{100}\mu(A)$, thus in particular $\frac{110}{100}\mu(A) > \mu(B) > \frac{90}{100}\mu(A)$. If $b \notin D(\Gamma, X, \mu)$ towards a contradiction, there is a b-invariant set $C \subseteq B$ in MALG_μ with $\mu(C) = \frac{1}{2}\mu(B)$. Then $\mu(\gamma^a(C)\Delta C) = \mu(\gamma^a(C)\Delta\gamma^b(C)) \leq 2^N\delta_{\Gamma,u}(a, b) < \frac{\epsilon^2}{100}\mu(A) < \frac{\epsilon^2}{90}\mu(B) = \frac{\epsilon^2}{45}\mu(C)$, $\forall\gamma \in Q$. Thus there is a-invariant $D \in \mathrm{MALG}_\mu$ with $\mu(C\Delta D) < \frac{8}{45}\mu(C) = \frac{8}{90}\mu(B)$.

Then

$$\begin{aligned}\mu(D\setminus A) \leq & \ \mu(D\Delta C)+\mu(C\setminus A)\\ \leq & \ \mu(D\Delta C)+\mu(C\setminus B)+\mu(A\Delta B)\\ < & \ \frac{8}{90}\mu(B)+\frac{8}{100}\mu(A)\\ < & \ \frac{20}{100}\mu(A),\end{aligned}$$

and $\mu(D)\geq\mu(C)-\mu(C\Delta D)>\frac{1}{2}\mu(B)-\frac{8}{90}\mu(B)=\frac{37}{90}\mu(B)>\frac{36}{100}\mu(A)$. Also $\mu(D)\leq\mu(C)+\mu(D\Delta C)<\frac{1}{2}\mu(B)+\frac{8}{90}\mu(B)=\frac{53}{90}\mu(B)<\frac{65}{100}\mu(A)$. So $0<\mu(A\cap D)<\mu(A)$ and $A\cap D$ is a-invariant, which violates the ergodicity of $a|A$.

We can in fact determine exactly the uniform path component of every $a\in A(\Gamma,X,\mu)$, using similar arguments.

Theorem 15.19 (Kechris). *Let Γ be a countable group with property* (T). *For each $a\in A(\Gamma,X,\mu)$, let $D_a=\bigcup\{A\in \mathrm{MALG}_\mu : \mu(A)>0,\ A$ is a-invariant and $a|A$ is ergodic$\}$ be the "discrete" part of the ergodic decomposition of a. Then a,b are in the same uniform path connected component of $A(\Gamma,X,\mu)$ iff $\mu(D_a)=\mu(D_b)$ and $a|D_a\cong b|D_b$.*

Proof. By 15.17 we can find a', uniformly path connected to a, and b', uniformly path connected to b, so that $a|D_a=a'|D_a$ and a' is trivial on $X\setminus D_a$ and similarly for b,b'. So we can assume that we work with a,b for which $a|(X\setminus D_a), b|(X\setminus D_b)$ are trivial. If then $a|D_a\cong b|D_b$ and $\mu(D_a)=\mu(D_b)$, clearly a,b are conjugate, thus belong to the same uniform path connected component. Conversely, assume such a,b are uniformly path connected. Then an argument similar to the proof of 15.17 shows that there is a 1-1 correspondence between the set $\mathcal{D}_a=\{A\in\mathrm{MALG}_\mu : \mu(A)>0$, A is a-invariant and $a|A$ is ergodic$\}$ and the corresponding $\mathcal{D}_b$, say $A\mapsto B_A$, with $a|A\cong b|B_A$, from which it follows that $\mu(D_a)=\mu(D_b)$ and $a|D_a\cong b|D_b$. □

Remark. Using the last Remark of Section 14, **(A)**, it follows that 15.19 holds for *every* group Γ, if the uniform topology is replaced by the $\delta_{\Gamma,\infty}$-topology.

(D) As we have seen in 12.2, ii), for every countable group Γ which does not have property (T), the generic element of $(A(\Gamma,X,\mu),w)$ is ergodic. The next result asserts that when Γ is a countable infinite group which has property (T), then the generic element of $(A(\Gamma,X,\mu),w)$ has continuous ergodic decomposition.

Theorem 15.20 (Kechris). *Let Γ be an infinite countable group with property* (T). *Then $C(\Gamma,X,\mu)$ is dense G_δ in $(A(\Gamma,X,\mu),w)$.*

Proof. By 15.14 it is enough to show that $C(\Gamma,X,\mu)$ is G_δ. Fix a Kazhdan pair (Q,ϵ) for Γ. Fix also a countable dense set $\mathcal{D}\subseteq\mathrm{MALG}_\mu\setminus\{\emptyset\}$. Then we claim that the following are equivalent for $a\in A(\Gamma,X,\mu)$:

(i) $a \in C(\Gamma, X, \mu)$,

(ii) For every $A \in \mathcal{D}$ and for all $n > 8$, we have that the condition $\forall \gamma \in Q(\mu(\gamma^a(A)\Delta A) < \frac{\epsilon^2}{n}\mu(A))$ implies that there is B such that $\mu(B \setminus A) < \frac{8}{n}\mu(A)$ and $(\frac{1}{2} - \frac{4}{n})\mu(A) < \mu(B) < (\frac{1}{2} + \frac{4}{n})\mu(A)$ and $\forall \gamma \in Q(\mu(\gamma^a(B)\Delta B) < \frac{\epsilon^2}{n}\mu(B))$,

This clearly shows that $C(\Gamma, X, \mu)$ is G_δ.

(i) $\Rightarrow$ (ii): Fix $A \in \mathcal{D}, n > 8$ such that $\forall \gamma \in Q(\mu(\gamma^a(A)\Delta A) < \frac{\epsilon^2}{n}\mu(A))$. Then by 15.18 find a-invariant $A' \in \mathrm{MALG}_\mu$ with $\mu(A\Delta A') < \frac{8}{n}\mu(A)$. Since $a \in C(\Gamma, X, \mu)$, there is a-invariant $B \in \mathrm{MALG}_\mu$, with $B \subseteq A'$ and $\mu(B) = \frac{1}{2}\mu(A')$. Then we have $\mu(B \setminus A) \leq \mu(A\Delta A') + \mu(B \setminus A') < \frac{8}{n}\mu(A)$ and $\mu(A) - \mu(A\Delta A') \leq \mu(A') \leq \mu(A) + \mu(A\Delta A')$, so $(1 - \frac{8}{n})\mu(A) < \mu(A') < (1 + \frac{8}{n})\mu(A)$, therefore $(\frac{1}{2} - \frac{4}{n})\mu(A) < \mu(B) < (\frac{1}{2} + \frac{4}{n})\mu(A)$. Obviously $\forall \gamma \in Q(\mu(\gamma^a(B)\Delta B) < \frac{\epsilon^2}{n}\mu(B))$.

(ii) $\Rightarrow$ (i): Fix $A' \in \mathrm{MALG}_\mu \setminus \{\emptyset\}$ which is a-invariant. We will find a-invariant $B' \in \mathrm{MALG}_\mu$ such that $0 < \mu(A' \cap B') < \mu(A')$. To do this, first choose a very large n (to satisfy what will be needed below). Then find $A \in \mathcal{D}$ so that $\mu(A\Delta A')$ is very small compared with $\mu(A')$ (and thus $\mu(A)$), so that, in particular, $\forall \gamma \in Q(\mu(\gamma^a(A)\Delta A) < \frac{\epsilon^2}{n}\mu(A))$. By (ii) then there is B with $\mu(B \setminus A) < \frac{8}{n}\mu(A), (\frac{1}{2} - \frac{4}{n})\mu(A) < \mu(B) < (\frac{1}{2} + \frac{4}{n})\mu(A)$ and $\forall \gamma \in Q(\mu(\gamma^a(B)\Delta B) < \frac{\epsilon^2}{n}\mu(B))$. Finally by 15.18 find a-invariant B' with $\mu(B\Delta B') < \frac{8}{n}\mu(B)$. It follows that $\mu(B' \setminus A')$ is very small compared to $\mu(A')$ and also $\mu(B')$ is very close to $\frac{1}{2}\mu(A')$, which implies that $0 < \mu(A' \cap B') < \mu(A')$ and completes the proof. $\square$

Remark. Recall that for Γ with property (T), Glasner-Weiss have shown that $\mathrm{ESIM}(\Gamma)$ is closed in $\mathrm{SIM}(\Gamma)$ (see the remarks following 12.2). The preceding result implies, in view of the result of Glasner-King (see Section 10, **(B)**), that the set of Borel probability measures in $\mathrm{ESIM}(\Gamma)$ which are continuous (i.e., non-atomic) is dense G_δ in the compact space of all probability measures on $\mathrm{ESIM}(\Gamma)$, which implies that $\mathrm{ESIM}(\Gamma)$ is perfect. We are using here the following result of the Choquet theory: if $P(\mathrm{ESIM}(\Gamma))$ is the compact metrizable space of the probability measures on $\mathrm{ESIM}(\Gamma)$, with the weak*-topology, the barycenter map $\mu \in P(\mathrm{ESIM}(\Gamma)) \mapsto b(\mu) \in \mathrm{SIM}(\Gamma)$ given by $\int f db(\mu) = \int(\int f de)d\mu(e)$, for any $f \in C(\mathbb{T}^\Gamma)$, is a homeomorphism of $P(\mathrm{ESIM}(\Gamma))$ and $\mathrm{SIM}(\Gamma)$, and for $\nu \in \mathrm{SIM}(\Gamma)$, the unique μ with $b(\mu) = \nu$ is the measure corresponding to the ergodic decomposition of ν with respect to the shift on $\mathbb{T}^\Gamma$.

Remark. From 15.20 and 12.2 it follows that for *any* countable group Γ the set $D(\Gamma, X, \mu) \setminus \mathrm{ERG}(\Gamma, X, \mu)$ is meager in $(A(\Gamma, X, \mu), w)$.

16. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$

(A) Gaboriau and Popa [GPo] were the first to construct continuum many non-OE measure preserving, free, ergodic actions of F_2, using methods from operator algebras. Later Törnquist [To1] used some ideas in the work of Popa [Po2] and Gaboriau-Popa together with a genericity argument to find a more elementary proof and established a descriptive strengthening of their result, which we will discuss in Section 17 (in a somewhat stronger form). A basic idea, going back to Popa, is to employ the standard action of $\mathrm{SL}_2(\mathbb{Z})$ (and its free subgroups of finite index) on $\mathbb{T}^2$. We will first recast below some important properties of this action in a somewhat different form that brings out more clearly its essential features. We will then derive in the next section the corollaries concerning orbit equivalence.

(B) Consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on $(X, \mu) = (\mathbb{T}^2, \mu)$, where μ is the usual product measure on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by the following matrix multiplication:

$$A \cdot (z_1, , z_2) = (A^{-1})^t \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Fix also a copy of F_2 in $\mathrm{SL}_2(\mathbb{Z})$ which has finite index in $\mathrm{SL}_2(\mathbb{Z})$, and consider the induced action a_0 of F_2 on (X, μ). This is free, measure preserving, ergodic (in fact weak mixing and even tempered (but not mixing), see Kechris [Kec4] 5, (C)). Denote by

$$E_{a_0} = F_0$$

the equivalence relation induced by this action. We will see that F_0 has the property that *all* its measure preserving extensions $E \supseteq F_0$ exhibit very interesting properties.

To formulate this, define the following left-invariant metric on $\mathrm{Aut}(X, \mu)$:

$$\begin{aligned} d_0(S, T) &= \delta_w(S, T) + \sum_n 2^{-n} \delta_u(S\gamma_n S^{-1}, T\gamma_n T^{-1}) \\ &= \delta_w(S, T) + \delta_{F_2, u}(S a_0 S^{-1}, T a_0 T^{-1}), \end{aligned}$$

where $F_2 = \{\gamma_n\}$ and $\gamma \in F_2$ is identified here with γ^{a_0}. Note that if $E, F \supseteq F_0$ are measure preserving countable Borel equivalence relations, and $T : E \cong F$ is an isomorphism, then $T\gamma T^{-1} \in [F]$, so if $N[E, F]$ denotes the set of all $T : E \cong F$, then for each such $F \supseteq F_0, d_0 | \bigcup\{N[E, F] : E \cong F, E \supseteq F_0\}$ is separable.

We can now recast the crucial properties of this action in the following lemma.

Theorem 16.1 (Popa). *For some $\delta > 0$ and any $E, F \supseteq F_0, T \in N[E, F]$, we have:*

$$d_0(T, 1) < \delta \Rightarrow \delta_u(T, 1) < 1 \Rightarrow E = F.$$

This has the following immediate corollary, noticing that if $E = F$,

$$N[E, F] = N[E]$$

and $d_0 | N[E]$ is continuous in the topology of $N[E]$.

Corollary 16.2. *For any $E \supseteq F_0, [E]$ is clopen in $N[E]$ and thus* $\mathrm{Out}(E)$ *is countable.*

Proof. Taking $E = F$ in 16.1, we see that there is $\epsilon > 0$ such that if $T \in N[E]$ and $d_{N[E]}(T,1) < \epsilon$, then $d_0(T,1) < \delta$, so $\delta_u(T,1) < 1$, i.e., $T(x) = x$ on a set of positive measure, thus (since clearly E is ergodic) $T \in [E]$. So $[E]$ is clopen in $N[E]$. □

Corollary 16.3. *On the set of measure preserving countable Borel equivalence relations $E \supseteq F_0$, isomorphism, $E \cong F$, is a countable equivalence relation (i.e., every equivalence class is countable).*

Proof. Fix $E \supseteq F_0$ and assume, towards a contradiction, that $\{E_i\}_{i\in I}$ is an uncountable family of distinct equivalence relations with $E_i \supseteq F_0$ and $E \cong E_i$. Fix isomorphisms $T_i : E_i \cong E$. Since $d_0|\bigcup_{F\cong E, F\supseteq F_0} N[F,E]$ is separable, it follows that $d_0|\{T_i : i \in I\}$ is separable, so there are $i \neq j$ in I with $d_0(T_i,T_j) < \delta$ (where δ is as in 16.1). Thus $d_0(T_i^{-1}T_j,1) < \delta$ and, since $T_i^{-1}T_j \in N[E_j,E_i]$, this implies, by 16.1, that $E_j = E_i$, a contradiction. □

Proof of 16.1. First we prove the implication

$$\delta_u(T,1) < 1 \Rightarrow E = F.$$

If $\delta_u(T,1) < 1$, then if $A = \{x : T(x) = x\}, \mu(A) > 0$. Thus the F_2-saturation of A is equal to X. Suppose that xEy. Then for some $\gamma,\delta \in F_2$, $\gamma\cdot x, \delta\cdot y \in A$, and $(\gamma\cdot x)E(\delta\cdot y)$, as $F_0 \subseteq E$. Then $T(\gamma\cdot x) = (\gamma\cdot x)F(\delta\cdot y) = T(\delta\cdot y)$, so xFy, as $F_0 \subseteq F$. Thus $E \subseteq F$ and similarly $F \subseteq E$.

We will now prove that there is δ such that

$$d_0(T,1) < \delta \Rightarrow \delta_u(T,1) < 1.$$

Below we will need some facts concerning the so-called *relative property* (T) for pairs of groups. These can be all found in [BdlHV], 1.4 and Jolissaint [Jo3].

The crucial point is that if we consider the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ by matrix multiplication and the corresponding semi-direct product $G_1 = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, then $(G_1,\mathbb{Z}^2)$ has the relative property (T), i.e., there is finite $Q \subseteq G_1$ and $\epsilon > 0$ such that in any unitary representation π of G_1 on a Hilbert space H, if there is a (Q,ϵ)-invariant vector v, i.e., $\|\pi(\gamma)(v) - v\| < \epsilon\|v\|, \forall\gamma \in Q$, then there is $\mathbb{Z}^2$-invariant non-0 vector. If $G = F_2 \ltimes \mathbb{Z}^2$, then $(G,\mathbb{Z}^2)$ also has the relative property (T) and by standard facts this also implies the following, which is all we will use in the sequel:

(*) There is finite $Q \subseteq F_2 \ltimes \mathbb{Z}^2$ and $\epsilon > 0$ such that for any unitary representation π of $F_2 \ltimes \mathbb{Z}^2$ on a Hilbert space H, if ξ is a (Q,ϵ)-invariant unit vector, then there is a $\mathbb{Z}^2$-invariant vector η with $\|\xi - \eta\| < 1$.

We will identify below $\mathbb{Z}^2$ with the group of characters $\widehat{\mathbb{T}^2}$ of $\mathbb{T}^2, \mathbf{m} = (m_1,m_2) \in \mathbb{Z}^2$ being identified with the character

$$\chi_{\mathbf{m}}(z_1,z_2) = z_1^{m_1} z_2^{m_2}$$

(where we view here $\mathbb{T}$ as the multiplicative group of the unit circle). Then the action of $\mathrm{SL}_2(\mathbb{Z})$ (and thus of $F_2 \leq \mathrm{SL}_2(\mathbb{Z})$) on $\mathbb{T}^2$ gives rise to an action on $\widehat{\mathbb{T}^2}$ given by $g \cdot \chi(z) = \chi(g^{-1} \cdot z)$ and by the above identification this is exactly the matrix multiplication action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. Thus the semidirect product $F_2 \ltimes \mathbb{Z}^2$ can be viewed (see Appendix I, **(B)**) as the set of pairs (γ, χ), with $\gamma \in F_2, \chi \in \widehat{\mathbb{T}^2}$ and multiplication defined by

$$(\gamma_1, \chi_1)(\gamma_2, \chi_2) = (\gamma_1\gamma_2, (\gamma_2^{-1} \cdot \chi_1)\chi_2).$$

Fix now $E, F \supseteq F_0, T \in N[E, F]$ and let (Q, ϵ) be as in (*) above. We will use T to define a unitary representation π of $F_2 \ltimes \mathbb{Z}^2$ on $H = L^2(F, M)$ (where M is defined as in Section 6, **(B)**). Letting $\pi(g)(f) = g \cdot f$, this is defined by

$$(\gamma, \chi) \cdot f(z, w) = \chi(\gamma^{-1}(z))\overline{\chi(\gamma^{-1}T^{-1}(w))}f(\gamma^{-1}(z), T\gamma^{-1}T^{-1}(w)),$$

for $f \in L^2(F, M)$, where again $\gamma \in F_2$ is identified with γ^{a_0}. (Note that this is well-defined as $zFw \Leftrightarrow \gamma^{-1}(z)FT\gamma^{-1}T^{-1}(w)$.) It is straightforward to check that this is an action.

Claim. *We can find $\delta > 0$ such that if $d_0(T, 1) < \delta$, then $\xi = \Delta =$ (the characteristic function of the diagonal) is (Q, ϵ)-invariant.*

Granting this, there is a $\mathbb{Z}^2$-invariant vector η in $L^2(F, M)$ with $\|\xi - \eta\| < 1$ and thus $\eta(z, z) \neq 0$ for a positive set of z, say A. Since $(1, \chi) \cdot \eta = \eta$ for all $\chi \in \widehat{\mathbb{T}^2}$, we have

$$\eta(z, z) = \chi(z)\overline{\chi(T^{-1}(z))}\eta(z, z),$$

thus

$$\chi(T^{-1}(z)) = \chi(z), \forall z \in A, \forall \chi \in \widehat{\mathbb{T}^2},$$

so, as the characters separate points, $T^{-1}(z) = z$ and so $T(z) = z$ on A, i.e., $\delta_u(T, 1) < 1$.

Proof of the claim. We have

$$(\gamma, \chi) \cdot \Delta(z, w) = \chi(\gamma^{-1}(z))\overline{\chi(\gamma^{-1}T^{-1}(w))}\Delta(\gamma^{-1}(z), T\gamma^{-1}T^{-1}(w)),$$

thus $\|(\gamma, \chi) \cdot \Delta - \Delta\|_2^2$ is equal to

$$\int_F \left|\chi(\gamma^{-1}(z))\overline{\chi(\gamma^{-1}T^{-1}(w))}\Delta(\gamma^{-1}(z), T\gamma^{-1}T^{-1}(w)) - \Delta(z, w)\right|^2 dM$$

and thus to

$$(1) \qquad \int_{\mathbb{T}^2} \left|\chi(\gamma^{-1}(z))\overline{\chi(\gamma^{-1}T^{-1}(z))}\Delta(\gamma^{-1}(z), T\gamma^{-1}T^{-1}(z)) - 1\right|^2 d\mu(z)$$

$$(2) \qquad + \int_{z \neq w, (z,w) \in F} \Delta(\gamma^{-1}(z), T\gamma^{-1}T^{-1}(w))dM.$$

Now (2) is equal to

$$\delta_u(\gamma^{-1}, T\gamma^{-1}T^{-1}) = \delta_u(\gamma, T\gamma T^{-1}),$$

so it is clear that if δ is small enough and $d_0(T,1) < \delta$, then (2)$< \frac{\epsilon^2}{2}$ for all $(\gamma,\chi) \in Q$. Also note that for (1) (considering cases whether $\gamma^{-1}(z) = T\gamma^{-1}T^{-1}(z)$ or not), we see that

$$(1) \leq \delta_u(\gamma^{-1}, T\gamma^{-1}T^{-1}) + \|(\gamma\cdot\chi) - U_T(\gamma\cdot\chi)\|^2,$$

where $\gamma\cdot\chi(z) = \gamma(\chi^{-1}(z))$ and U_T is the unitary operator on $L^2(\mathbb{T}^2,\mu)$ corresponding to T (i.e., $U_T(f)(z) = f(T^{-1}(z))$. Now $d_0(T,1) \geq \delta_w(T,1)$ and the topology induced by δ_w is the weak topology, i.e., the topology induced by identifying T and U_T and using on $U(L^2(\mathbb{T}^2,\mu))$ the strong operator topology. So if δ is small enough, we have again $d_0(T,1) < \delta \Rightarrow$ (1) $< \frac{\epsilon^2}{2}, \forall(\gamma,\chi) \in Q$, and thus finally

$$d_0(T,1) < \delta \Rightarrow \|(\gamma,\chi)\cdot\Delta - \Delta\|_2 < \epsilon,$$

for all $(\gamma,\chi) \in Q$ and the proof is complete. □

17. Non-orbit equivalent actions of free groups

(A) We will now prove the following non-classification result. The first assertion is in Törnquist [To1] and the second can be obtained by combining his method with 5.5.

Theorem 17.1 (Törnquist). *(a) The equivalence relation E_0 can be Borel reduced to the orbit equivalence relation of measure preserving, free, ergodic actions of the free group F_2.*

(b) This orbit equivalence relation cannot be classified by countable structures.

Proof. We fix, by the work in Section 16, a measure preserving, free, ergodic action a_0 of F_2 on (X,μ) such that if $F_0 = E_{a_0}$ is the induced equivalence relation, then isomorphism on the set of all $E \supseteq F_0$ is a countable equivalence relation. Let $F_2 = \langle\alpha,\beta\rangle, F_3 = \langle\alpha,\beta,\gamma\rangle$ be free generators.

Lemma 17.2. *Fix $a_0 \in \mathrm{FR}(F_2,X,\mu)$. Then for a comeager set of $S \in \mathrm{Aut}(X,\mu)$ the action a_S of F_3 determined by $\alpha^{a_S} = \alpha^{a_0}, \beta^{a_S} = \beta^{a_0}, \gamma^{a_S} = S$ is free (and of course ergodic).*

We will grant this and complete the proof of 17.1. We will first prove the version of 17.1 for F_3 instead of F_2. Consider the equivalence relation $\sim$ on $\mathrm{Aut}(X,\mu)$ defined by

$$S \sim T \Leftrightarrow a_S \mathrm{OE} a_T.$$

To prove (b), note that if orbit equivalence of measure preserving, free, ergodic actions of F_3 can be classified by countable structures, so can $\sim$ and we will derive a contradiction from this.

From 5.5, we know that the conjugacy action of $([F_0],u)$ on the space $(\mathrm{Aut}(X,\mu),w)$ is generically turbulent. Let $\sim_C$ be the corresponding equivalence relation. Then clearly

$$S \sim_C T \Rightarrow E_{a_S} = E_{a_T} \Rightarrow a_S \mathrm{OE} a_T \Leftrightarrow S \sim T.$$

So if $f : \mathrm{Aut}(X, \mu) \to X_L$, f Borel and L some countable language, is such that

$$S \sim T \Leftrightarrow f(S) \cong f(T),$$

then

$$S \sim_C T \Rightarrow f(S) \cong f(T),$$

so, by turbulence, there is a comeager set A and $M_0 \in X_L$ with

$$f(S) \cong M_0, \forall S \in A.$$

Now we claim that every $\sim$-class is meager, which implies that there are $S, T \in A$ with $S \not\sim T$ and thus $f(S) \ncong f(T)$, a contradiction. To prove this let

$$S \approx T \Leftrightarrow E_{a_S} = E_{a_T}.$$

Then by the above property of F_0 (since $E_{a_S}, E_{a_T} \supseteq F_0$), we have that each $\sim$-class contains only countably many $\approx$-classes, so it is enough to check that each $\approx$-class is meager. But this is obvious, as $[S]_\approx \subseteq [E_{a_S}]$, which is a meager set in $(\mathrm{Aut}(X, \mu), w)$.

To prove (a) it is enough to show that E_0 is Borel reducible to $\sim$. Now $\sim$ is a meager equivalence relation (being analytic with meager classes) and contains the equivalence relation induced by the conjugacy action of $([F_0], u)$ on $(\mathrm{Aut}(X, \mu), w)$, which clearly has a dense orbit. So E_0 is Borel reducible to $\sim$ by the argument in Becker-Kechris [BK], 3.4.5 (see also Hjorth-Kechris [HK1], 3.2).

To prove the result for F_2, it is enough to show that OE for measure preserving, free, ergodic actions of F_3 can be Borel reduced to OE for measure preserving, free, ergodic actions of F_2. To see this, fix (X, μ) and let $Y = X \times \{0, 1\}, \nu = \mu \times p$, where $p(\{0\}) = p(\{1\}) = 1/2$. Let $\tau \in \mathrm{Aut}(Y, \nu)$ be defined by $\tau(x, i) = (x, 1-i)$. Each $S \in \mathrm{Aut}(X, \mu)$ induces $\tilde{S} \in \mathrm{Aut}(Y, \nu)$ by $\tilde{S}(x, 0) = (S(x), 0), \tilde{S}(x, 1) = (x, 1)$. Given $a \in \mathrm{FRERG}(F_3, X, \mu)$, let $\tilde{E}_a$ be the equivalence relation on Y induced by $\tilde{\alpha^a}, \tilde{\beta^a}, \tilde{\gamma^a}, \tau$, where $F_3 = \langle \alpha, \beta, \gamma \rangle$. Then it is not hard to see that

$$E_a \cong E_b \Leftrightarrow \tilde{E}_a \cong \tilde{E}_b.$$

Clearly $\tilde{E}_a$ is ergodic. Finally, it is also not hard to see that $\tilde{E}_a$ is induced by a free, measure preserving action $\tilde{a}$ of $F_2 = \langle \alpha, \beta \rangle$, namely

$$\alpha^{\tilde{a}}(x, i) = \begin{cases} \tau(x, i), & \text{if } i = 0, \\ \widetilde{\alpha^a}\tau(x, i), & \text{if } i = 1, \end{cases}$$

$$\beta^{\tilde{a}}(x, i) = \begin{cases} \tau\widetilde{\gamma^a}(x, i), & \text{if } i = 0, \\ \widetilde{\beta^a}\tau(x, i), & \text{if } i = 1. \end{cases}$$

So it only remains to give the proof of 17.2.

Proof of 17.2. Let $T_1 = \alpha^{a_0}, T_2 = \beta^{a_0}$. For each $\epsilon > 0$, and non-trivial reduced word $v \in F_3$, the set

$$\{S : \delta_u(v^{T_1, T_2, S}, 1) > 1 - \epsilon\}$$

is open in $(\mathrm{Aut}(X, \mu), w)$, so, by the Baire Category Theorem, it is enough to show that it is dense. By induction, we can assume that the intersection of the following sets, where $\bar{v}$ varies over all words of length less than that of v,

$$\{S : \delta_u(\bar{v}^{T_1,T_2,S}, 1) = 1\},$$

is dense in $(\mathrm{Aut}(X, \mu), w)$. Put for convenience for each word ω,

$$\omega^S = \omega^{T_1,T_2,S}.$$

Finally, we can assume that γ occurs in v. Let $v_0, v_1, v_2, \dots, v_n = v$, where $n = \mathrm{length}(v)$, be the terminal subwords of v with $\mathrm{length}(v_i) = i$, and let $i_0 < n$ be largest such that $v_{i_0+1} = \gamma^{\pm 1} v_{i_0}$. Assume, without loss of generality that $v_{i_0+1} = \gamma v_{i_0}$, so that $v_{i_0+1}^S = S v_{i_0}^S$.

We will use the following two simple lemmas.

Lemma 17.3. *If $P_1, \dots, P_k \in \mathrm{Aut}(X, \mu)$ and*

$$\mu(\{x : P_1(x), \dots, P_k(x) \text{ are distinct}\}\ ) > a,$$

then there are pairwise disjoint Borel sets $A_1, \dots, A_m$ with $\mu(A_1 \cup \dots \cup A_m) > a$ and $P_1(A_i), \dots, P_k(A_i)$ pairwise disjoint, for each $1 \leq i \leq m$.

Proof. As in the proof of 3.10, there is a countable collection of Borel sets $\{U_n\}$ such that for each x with $P_1(x), \dots, P_k(x)$ distinct, there is $U_{n(x)}$ with $x \in U_{n(x)}$ and $P_1(U_{n(x)}), \dots, P_k(U_{n(x)})$ pairwise disjoint. Thus, if $A = \{x : P_1(x), \dots, P_k(x) \text{ are distinct}\}$, then $A = \bigcup_{i \in \mathbb{N}} A_i$, where the sets $P_1(A_i), \dots, P_k(A_i)$ are pairwise disjoint, from which the conclusion follows. □

Lemma 17.4. *For any $T \in \mathrm{Aut}(X, \mu)$ and any Borel sets $A, A_1, \dots, A_m$, there is $S \in \mathrm{Aut}(X, \mu)$, such that $S(A_i) = T(A_i), S^{-1}(A_i) = T^{-1}(A_i)\ \forall i \leq m$, and $\{x : S(x) \neq T(x)\} = A$.*

Proof. Considering the partition generated by

$$A, A_1, \dots, A_m, T^{-1}(A_1), \dots, T^{-1}(A_m)$$

and the fact that for each Borel set B there is an involution $P_B \in \mathrm{Aut}(X, \mu)$ with $\mathrm{supp}(P_B) = B$, it follows that there is an involution $P \in \mathrm{Aut}(X, \mu)$ with $\mathrm{supp}(P) = A$ and $P(A_i) = A_i, P(T^{-1}(A_i)) = T^{-1}(A_i), \forall i \leq m$. Let then $S = TP$. □

Continuing the proof, it is enough to show that for each S_0 such that $\delta_u(\bar{v}^{S_0}, 1) = 1$ for every word $\bar{v}$ of length $< n$, and each weak nbhd N_0 of S_0,

$$N_0 \cap \{S : \delta_u(v^S, 1) > 1 - \epsilon\} \neq \emptyset.$$

By 17.3, we can find pairwise disjoint Borel sets $A_1, \dots, A_m \subseteq X$ with $\mu(\bigcup_{i=1}^m A_i) > 1 - \epsilon$ and

$$v_i^{S_0}(A_i) \cap v_j^{S_0}(A_i) = \emptyset, i \neq j < n.$$

If $\delta_u(v^{S_0}, 1) > 1 - \epsilon$, there is nothing to prove. Else (renumbering the A_i's if necessary) on a set of positive measure in A_1 we have $v^{S_0}(x) = x$.

Put $A_1' = \{x \in A_1 : v^{S_0}(x) = x\}$. Then, by 17.4, find $S_1 \in N_0$ such that $\{x : S_1(x) \neq S_0(x)\} = v_{i_0}^{S_0}(A_1')$. It follows that $v^{S_1}(x) \neq x$ for $x \in A_1$. (Note that $S_1 = S_0, S_1^{-1} = S_0^{-1}$ on the sets $A_1, v_1^{S_0}(A_1), \dots, v_{i_0-1}^{S_0}(A_1)$.) Then, by Lemma 17.3 again, we can find pairwise disjoint $B_1, \dots, B_k \subseteq A_1$ such that $\mu(\bigcup_{j=1}^k B_j \cup A_2 \cup \dots \cup A_m) > 1 - \epsilon$ and $v^{S_1}(B_j) \cap B_j = \emptyset, \forall j \leq k$.

If $\delta_u(v^{S_1}, 1) > 1-\epsilon$, then again there is nothing to prove, so (renumbering again the A_i's, $i \geq 2$, if necessary), there is a set of positive measure in A_2 for which $v^{S_1}(x) = x$. Put $A_2' = \{x \in A_2 : v^{S_1}(x) = x\}$. As before we can find $S_2 \in N_0$ such that $\{x : S_2(x) \neq S_1(x)\} = v_{i_0}^{S_1}(A_2')$ and $S_2^{\pm 1}(v_i^{S_1}(B_j)) = S_1^{\pm 1}(v_i^{S_1}(B_j)), \forall i < n, j \leq k$, so that $v_i^{S_2}(B_j) = v_i^{S_1}(B_j), \forall i \leq n,\ j \leq k$, thus $v^{S_2}(B_j) \cap B_j = \emptyset, \forall j \leq k$. Also, as before, $v^{S_2}(x) \neq x$ for $x \in A_2$, so we can find pairwise disjoint $C_1, \dots, C_\ell \subseteq A_2$ with $\mu(\bigcup_{j=1}^k B_j \cup \bigcup_{t=1}^\ell C_t \cup A_3 \cup \dots \cup A_m) > 1-\epsilon$ and $v^{S_2}(B_j) \cap B_j = v^{S_2}(C_t) \cap C_t = \emptyset, \forall j \leq k, \forall t \leq \ell$. Proceeding this way, finitely many times, we can find $\bar{S} \in N_0$ with $\delta_u(v^{\bar{S}}, 1) > 1 - \epsilon$ and the proof is complete. □

(B) In connection with 17.1, we note that it would also follow from 13.5 if one knew that orbit equivalence classes in $\mathrm{ERG}(F_2, X, \mu)$ were meager in $(\mathrm{ERG}(F_2, X, \mu), w)$. So we can raise the following general question (analogous to that of 13.6 for orbit equivalence).

Problem 17.5. *For which infinite countable groups* Γ *are the orbit equivalence classes in* $\mathrm{ERG}(\Gamma, X, \mu)$ *meager in the weak topology of* $\mathrm{ERG}(\Gamma, X, \mu)$*?*

Dye's theorem 3.13 and the Ornstein-Weiss [OW] theorem that every measure preserving action of a countable amenable group gives rise to a hyperfinite equivalence relation, imply that there is only one orbit equivalence class in $\mathrm{ERG}(\Gamma, X, \mu)$, when Γ is infinite amenable, so 17.5 has clearly a negative answer for Γ amenable.

Finally, in view of 14.4 and 17.1, we have the following problem.

Problem 17.6. *Let* Γ *be a non-amenable group. Is it true that orbit equivalence of free, measure preserving, ergodic actions of* Γ *cannot be classified by countable structures?*

Addendum. Recently Adrian Ioana [I1] has shown that any group Γ with $F_2 \leq \Gamma$ has continuum many non-orbit equivalent, free, measure preserving, ergodic actions. In fact using his main lemma one can derive a positive answer to 17.6 for such groups, and also show that E_0 can be Borel reduced to orbit equivalence.

We outline the argument below. Consider the action $a_0 \in A(F_2, X, \mu)$, where $X = \mathbb{T}^2, \mu =$ the usual product measure on $\mathbb{T}^2$, defined in Section 16, (**B**). This action is free and weak mixing. Let also Γ be a countable group with $F_2 \leq \Gamma$.

The following is proved in Ioana [I1], 1.3:

Consider the set $\mathcal{A} \subseteq A(\Gamma, Y, \nu)$ ((Y, ν) some fixed space) of all $a \in \mathrm{FR}(\Gamma, Y, \nu)$ satisfying the following conditions:

(i) $a|F_2 \in \mathrm{ERG}(F_2, Y, \nu)$,

(ii) There is a Borel map $\varphi : Y \to X$ witnessing that a_0 is a factor of $a|F_2$ such that for each $\gamma \in \Gamma, \gamma \neq 1, \{y \in Y : \pi(y) = \pi(\gamma^a(y))\}$ is ν-null.

Define the equivalence relation R on $\mathcal{A}$ by: $aRb \Leftrightarrow a|F_2 \cong b|F_2$. Then on the set $\mathcal{A}$, OE has countable index over OE$\cap R$ (i.e., every OE class in $\mathcal{A}$ contains only countably many OE$\cap R$ classes).

One can also see, using, for example, Ioana [I1], 2.2 (i) that $\mathrm{CInd}_{F_2}^{\Gamma}(a_0)|F_2$ is weak mixing. Let now $\pi \in \mathrm{Irr}(F_2, H), \pi \prec \lambda_{F_2}$ (see Appendix H). Let $\tilde{\pi} = \mathrm{Ind}_{F_2}^{\Gamma}(\pi)$ be the induced representation. Since $\mathrm{Ind}_{F_2}^{\Gamma}(\lambda_{F_2}) \cong \lambda_\Gamma$ and $\pi \prec \lambda_{F_2} \Rightarrow \mathrm{Ind}_{F_2}^{\Gamma}(\pi) \prec \mathrm{Ind}_{F_2}^{\Gamma}(\lambda_{F_2})$, it follows that $\tilde{\pi} \prec \lambda_\Gamma$ (see Bekka-de la Harpe-Valette [BdlHV]). Moreover $\tilde{\pi}|\Delta \prec \lambda_\Gamma|\Delta \sim \lambda_\Delta$, for any $\Delta \leq \Gamma$, so $\tilde{\pi}|F_2 \prec \lambda_{F_2}$ and thus $\tilde{\pi}|F_2$ is weak mixing (since λ_{F_2} does not weakly contain a finite-dimensional representation; see Dixmier [Di], 18.9.5, 18.3.6). Let now $a_{\tilde{\pi}}$ be as in Theorem E.1, so that $a_{\tilde{\pi}}|F_2 \in \mathrm{WMIX}(F_2, X, \mu)$ (see the paragraph following E.1). Put

$$b(\pi) = \mathrm{CInd}_{F_2}^{\Gamma}(a_0) \times a_{\tilde{\pi}}.$$

Then, recalling Appendix H, (**C**), $b : S(\lambda_{F_2}) \to \mathrm{FR}(\Gamma, Y, \nu)$ (for some appropriate (Y, ν)) is a Borel map (where $A(\Gamma, Y, \nu)$ has the weak topology) and $b(\pi) \in \mathcal{A}$.

Thus $b^{-1}(\mathrm{OE})$ has countable index over $b^{-1}(\mathrm{OE} \cap R)$ and we next verify that $b^{-1}(\mathrm{OE} \cap R)$ has countable index over $\cong$. Indeed, if $\pi, \rho \in S(\lambda_{F_2})$ and $b(\pi)Rb(\rho)$, then $\rho \leq \tilde{\rho}|F_2 \leq \kappa_0^{a_{\tilde{\rho}}}|F_2 \leq \kappa_0^{b(\rho)}|F_2$ (as $a_{\tilde{\rho}}|F_2 \sqsubseteq b(\rho)|F_2$) $\cong \kappa_0^{b(\pi)}|F_2$. Thus fixing π, all such ρ are irreducible subrepresentations of $\kappa_0^{b(\pi)}|F_2$, so there are only countably many, up to $\cong$.

It follows that $b^{-1}(\mathrm{OE})$ has countable index over $\cong$ in the space $S(\lambda_{F_2})$. Now the conjugacy action of $U(H)$ on $S(\lambda_{F_2})$ is turbulent by Appendix H, (**C**) (see the second to the last remark). As in the proof of 17.1, this shows that E_0 can be Borel reduced to OE in $\mathrm{FRERG}(\Gamma, X, \mu)$ and also that the latter cannot be classified by countable structures.

Addendum. Inessa Epstein [E] has now shown that *any* non-amenable group Γ has continuum many non-orbit equivalent, free, measure preserving, ergodic actions. By combining her work with that of Ioana-Kechris-Tsankov [IKT] one can now obtain a complete solution to Problem 17.6. We have in fact the following stronger result.

Theorem 17.7 (Epstein-Ioana-Kechris-Tsankov; see [IKT]). *Let Γ be a non-amenable group. Then E_0 can be Borel reduced to the orbit equivalence relation of measure preserving, free, mixing actions of Γ and this orbit equivalence relation cannot be classified by countable structures.*

Thus we have a very strong dichotomy: If an infinite group Γ is amenable, there is exactly one orbit equivalence class of measure preserving, free, ergodic actions of Γ. If Γ is not amenable, orbit equivalence of such actions (even if they are mixing) is unclassifiable in a very strong sense.

18. Classifying group actions: A survey

In several places before we have considered various notions of equivalence of actions of a given group Γ and discussed a number of results concerning the complexity of classification under each notion of equivalence. Our aim in this section is to summarize what seems to be known in this context and collect problems that are left open.

Fix a countable infinite group Γ. In the space $A(\Gamma, X, \mu)$ we have considered the following equivalence relations:

(i) $a \cong b$ (isomorphism)
(ii) $a \mathrm{OE} b$ (orbit equivalence)
(iii) $a \cong^w b$ (weak isomorphism)
(iv) $\kappa_0^a \cong \kappa_0^b$ (unitary equivalence)
(v) $a \sim b$ (weak equivalence)

These are discussed in Section 10. It is clear that (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (v). To avoid trivialities, we will consider these equivalence relations on the space $\mathrm{FRERG}(\Gamma, X, \mu)$ of free, ergodic actions of Γ. We will start with $\Gamma = \mathbb{Z}$, then consider abelian Γ, amenable Γ and general (infinite) Γ, as the information available is decreasing as we pass from one case to the next. For each group Γ, we will consider relations between (i)–(v), the Borel complexity of each one of them (in the weak topology) and finally their complexity in the hierarchy of equivalence relations.

Case I. $\Gamma = \mathbb{Z}$.

(Ia) *Comparisons.* By Dye's Theorem, any two $a, b \in \mathrm{FRERG}(\mathbb{Z}, X, \mu) = \mathrm{ERG}(\mathbb{Z}, X, \mu) = \mathrm{ERG}$ are orbit equivalent, so clearly (ii) $\not\Rightarrow$ (i). It is known that (iii) $\not\Rightarrow$ (i); see Glasner [Gl2], 7.16. If $a \cong^w b$, then a, b have the same entropy, so, considering Bernoulli shifts of different entropy, we see that (iv) $\not\Rightarrow$ (iii). Finally, we have again that $a \sim b$ for any $a, b \in \mathrm{ERG}(\mathbb{Z}, X, \mu)$ (see 2.4), so (v) $\not\Rightarrow$ (iii).

(Ib) *Borel complexity.* Foreman-Rudolph-Weiss [FRW] have shown that $\cong$ on ERG is not Borel (it is clearly analytic). (ii), (v) are of course trivial on ERG and $\cong^w$ is analytic. **Open problem**: Is $\cong^w$ on ERG Borel? Finally, by spectral theory, (iv) is Borel.

(Ic) *Equivalence relation complexity.* In terms of Borel reducibility, a (strict) lower bound for $\cong$ is measure equivalence (see 5.7) and an obvious upper bound is the universal equivalence relation induced by a Borel action of the group $\mathrm{Aut}(X, \mu)$ (or equivalently induced by a Borel action of $U(H)$); see Becker-Kechris [BK] for a discussion of these universal equivalence relations. **Open problem**. Is $\cong$ on ERG Borel bireducible to the universal equivalence relation induced by a Borel action of $\mathrm{Aut}(X, \mu)$? We have seen in 5.7 that (iv) is Borel bireducible to measure equivalence and of course (ii), (v) are trivial. Finally $\cong^w$ is not classifiable by countable structures; see 13.9. **Open problem**. Does E_1 Borel reduce to $\cong^w$ on ERG? (See 13.10.)

Remark. One can also ask about the complexity of isomorphism within each unitary equivalence class of ergodic transformations. Not much seems to be known about this problem; see Section 5, **(C)**.

Case II. Γ abelian.

(IIa) *Comparisons.* Same as with $\Gamma = \mathbb{Z}$, except that I do not know if (iii) $\not\Rightarrow$ (i) has been verified for all abelian Γ.

(IIb) *Borel complexity.* Same as with $\Gamma = \mathbb{Z}$, except that we have an additional open problem. **Open problem**. Is $\cong$ on $\mathrm{FRERG}(\Gamma, X, \mu)$ non-Borel?

(IIc) *Equivalence relation complexity.* Same as with $\Gamma = \mathbb{Z}$.

Case III. Γ amenable.

(IIIa) *Comparisons.* Same as with Γ abelian.

(IIIb) *Borel complexity.* Same as with Γ abelian, except that we have an additional open problem. **Open problem**. Is unitary equivalence on $\mathrm{FRERG}(\Gamma, X, \mu)$ Borel?

(IIIc) *Equivalence relation complexity.* Concerning isomorphism in the space $\mathrm{FRERG}(\Gamma, X, \mu)$, we have seen in 13.7 that it is not classifiable by countable structures. Since this is true even allowing C-measurable functions, it follows from Hjorth-Kechris [HK1], 2.2 that E_0 can be Borel reduced to $\cong$ on $\mathrm{FRERG}(\Gamma, X, \mu)$. (Recall that a function between standard Borel spaces is *C-measurable* if the preimage of each Borel set is in the smallest σ-algebra containing the Borel sets and closed under the Souslin operation $\mathcal{A}$.) Not much else seems to be known about the equivalence relation complexity of $\cong$. Again OE is trivial and so is (v). Nothing seems to be known about weak isomorphism and unitary equivalence.

Case IV. General Γ.

(IVa) *Comparisons.* As far as I know, it is open whether for *every* infinite countable group Γ the reversals of the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (v) fail.

Addendum. Kida [Ki] provides examples of groups Γ for which (i) $\Leftrightarrow$ (ii). Also a recent result of Bowen [Bo] shows that for any $\Gamma \supseteq F_2$, (iii) $\Rightarrow$ (i) fails.

(IVb) *Borel complexity.* It is again open whether the relations (i)–(iv) are non-Borel for *every* countable group Γ. Clearly weak equivalence is G_δ.

Addendum. Törnquist [To3] has recently shown that isomorphism and orbit equivalence are not Borel for groups Γ with relative property (T) over an infinite subgroup.

(IVc) *Equivalence relation complexity.* As in (IIIc), $\cong$ on the space $\mathrm{FRERG}(\Gamma, X, \mu)$ cannot be classified by countable structures and E_0 can be Borel reduced to $\cong$ on $\mathrm{FRERG}(\Gamma, X, \mu)$. Concerning orbit equivalence, it is now known (see 17.7) that for non-amenable Γ, E_0 can be Borel reduced to OE on $\mathrm{FRERG}(\Gamma, X, \mu)$ and the latter is not classifiable by countable structures. Nothing much is known about weak isomorphism. Concerning unitary equivalence, we have seen in 13.8 that it cannot be classified by countable structures and again it follows that E_0 can be Borel reduced to it. Finally, weak equivalence, is G_δ and thus smooth. It does not seem to be known however whether for *every* non-amenable Γ, there is more than one weak equivalence class in $\mathrm{FRERG}(\Gamma, X, \mu)$ (see the fourth remark of Section 13, **(A)**).

CHAPTER III

Cocycles and cohomology

19. Group-valued random variables

Let G be a Polish group and $d_G = d$ a compatible metric bounded by 1. Let (X, μ) be a standard measure space. We denote by

$$L(X, \mu, G)$$

the space of all Borel (equivalently μ-measurable) maps $f : X \to G$, where we identify two such maps if they agree μ-a.e. Thus $L(X, \mu, G)$ is the space of G-valued random variables. We endow $L(X, \mu, G)$ with pointwise multiplication

$$fg(x) = f(x)g(x),$$

under which it is clearly a group. We next define a metric $d_{X,\mu,G}$ on $L(X, \mu, G)$ by

$$d_{X,\mu,G}(f, g) = \int d(f(x), g(x))d\mu(x).$$

Convention. *When G is a countable (discrete) group, we will always assume that it is equipped with the metric $d_G = d$, where $d(x, y) = 1$, if $x \neq y$.*

It follows that when G is countable,

$$d_{X,\mu,G}(f, g) = \mu(\{x : f(x) \neq g(x)\}).$$

The following proposition is obvious.

Proposition 19.1. *If d is left-invariant (resp., right-invariant, (two-sided) invariant), so is $d_{X,\mu,G}$.*

We denote by $\tau(d_{X,\mu,G})$ the topology on $L(X, \mu, G)$ induced by $d_{X,\mu,G}$.

Proposition 19.2. *$\tau(d_{X,\mu,G})$ is a group topology.*

Proof. To simplify the notation, write below $\tilde{d} = d_{X,\mu,G}$.

Assume $\tilde{d}(f_n, f) \to 0, \tilde{d}(g_n, g) \to 0$ in order to show that $\tilde{d}(f_n^{-1}, f^{-1}) \to 0, \tilde{d}(f_n g_n, fg) \to 0$. Put $r_n(x) = d(f_n(x), f(x)), s_n(x) = d(f_n(x)^{-1}, f(x)^{-1})$. Then $r_n \geq 0, r_n \in L^1(X, \mu, \mathbb{R}), r_n \to 0$ in $L^1(X, \mu, \mathbb{R})$, so $\exists \{n_i\}$ such that $r_{n_i}(x) \to 0$, μ-a.e. Thus $s_{n_i}(x) \to 0$ μ-a.e., so by Lebesgue Dominated Convergence $\tilde{d}(f_{n_i}^{-1}, f^{-1}) \to 0$. Similarly, for any sequence $\{m_i\}$ there is a subsequence $\{n_i\}$ with $\tilde{d}(f_{n_i}^{-1}, f^{-1}) \to 0$, so $\tilde{d}(f_n^{-1}, f^{-1}) \to 0$. The argument for multiplication is essentially the same. □

Proposition 19.3. *The topology $\tau(d_{X,\mu,G})$ depends only on the topology of G and the measure class of μ, i.e., if $d_1, d_2 \leq 1$ are compatible metrics on G and $\mu_1 \sim \mu_2$, then $\tau((d_1)_{X,\mu_1,G}) = \tau((d_2)_{X,\mu_2,G})$.*

Proof. First we show that for any μ, $\tau((d_1)_{X,\mu,G}) = \tau((d_2)_{X,\mu,G})$. Fix $\epsilon > 0$ and find $0 < \delta < \epsilon$ such that $B_{d_1}(1, \delta/2) \subseteq B_{d_2}(1, \epsilon/2)$. Then we claim that (in the notation of the proof of 19.2) $B_{\tilde{d}_1}(1, (\delta/2)^2) \subseteq B_{\tilde{d}_2}(1, \epsilon)$. Indeed, if $\tilde{d}_1(1, f) = \int d_1(1, f(x))d\mu(x) < (\delta/2)^2$, then, by Chebychev's inequality, $\mu(\{x : d_1(1, f(x)) \geq \delta/2\}) < \delta/2$, so $\tilde{d}_2(1, f) \leq \delta/2 + \delta/2 < \epsilon$.

Next we show that for any d, $\tau(d_{X,\mu_1,G}) = \tau(d_{X,\mu_2,G})$. Fix again $\epsilon > 0$ and find $0 < \delta < \epsilon$ such that for any Borel set B, $\mu_1(B) < \delta/2 \Rightarrow \mu_2(B) < \epsilon/2$. Then, as before, $B_{\tilde{d}_{\mu_1}}(1, (\delta/2)^2) \subseteq B_{\tilde{d}_{\mu_2}}(1, \epsilon)$ (where $\tilde{d}_{\mu_i} = d_{X,\mu_i,G}$). □

From now on we will simply write $\tau_{X,\mu,G}$ for this topology. We will next verify that it coincides with the topology of *convergence in measure.* Recall that for $f_n, f \in L(X, \mu, G), f_n \to f$ in measure if $\forall\epsilon > 0(\mu(\{x : d(f_n(x), f(x)) \geq \epsilon\}) \to 0)$.

Proposition 19.4. *The following are equivalent for $f_n, f \in L(X, \mu, G)$:*

(i) $f_n \to f$ (in $\tau_{X,\mu,G}$),

(ii) $f_n \to f$ in measure,

(iii) For every sequence $\{m_i\}$, there is a subsequence $\{n_i\}$ such that $f_{n_i}(x) \to f(x)$, μ-a.e. (x).

Proof. (i) $\Rightarrow$ (ii): By Chebychev's inequality, $\epsilon \cdot \mu(\{x : d(f(x), g(x)) \geq \epsilon\}) \leq d_{X,\mu,G}(f, g)$.

(ii) $\Rightarrow$ (i): Let $r_n(x) = d(f_n(x), f(x))$. If $f_n \to f$ in measure, then $r_n : X \to \mathbb{R}$ converges in measure to 0, so there is $\{n_i\}$ such that $r_{n_i} \to 0$ a.e., thus by Lebesgue Dominated Convergence, $d_{X,\mu,G}(f_{n_i}, f) \to 0$. Similarly for any sequence $\{m_i\}$, there is a subsequence $\{n_i\}$ such that $d_{X,\mu,G}(f_{n_i}, f) \to 0$, so $f_n \to f$.

This argument also shows that (ii) $\Rightarrow$ (iii). Finally, (iii) $\Rightarrow$ (i) follows immediately by Lebesgue Dominated Convergence. □

Since every two standard measure spaces $(X_1, \mu_1), (X_2, \mu_2)$ are isomorphic, it follows that $(L(X, \mu, G), \tau_{X,\mu,G})$ (resp., $(L(X, \mu, G), d_{X,\mu,G})$) is determined uniquely up to homeomorphism (resp., isometry). Thus, if we do not need to indicate (X, μ) explicitly, we will sometimes write

$$\tilde{G} = L(X, \mu, G),$$
$$\tilde{\tau} = \tau_{X,\mu,G},$$
$$\tilde{d} = d_{X,\mu,G}.$$

Next we note that the map $g \in G \mapsto (x \mapsto g)$ is an isometric embedding from G to the subgroup of constant functions in $\tilde{G}$, so we can view G as a (topological) subgroup of $\tilde{G}$, $G \leq \tilde{G}$, and $\tilde{d}$ as an extension of d.

Consider now for $n \geq 1$ the product group G^{2^n} with compatible metric

$$d_n(\{g_i\}, \{h_i\}) = 2^{-n} \sum_{i=1}^{2^n} d(g_i, h_i).$$

We can embed isometrically G^{2^n} into $G^{2^{n+1}}$ via

$$(g_1, \dots, g_{2^n}) \to (g_1, g_1, g_2, g_2, \dots, g_{2^n}, g_{2^n}),$$

so we can write $G^1 \leq G^2 \leq G^4 \leq \dots$ and let $G^{(\infty)} = \bigcup_n G^{(2^n)}$, which is a separable metrizable group with compatible metric $\bigcup_n d_n$.

Since (X, μ) is a standard measure space, we can find a sequence of finite Borel partitions of $X, \mathcal{P}_1 \geq \mathcal{P}_2 \geq \dots$, with $\mathcal{P}_{n+1}$ refining $\mathcal{P}_n, \mathcal{P}_n = \{A_1^{(n)}, \dots, A_{2^n}^{(n)}\}, \mu(A_i^{(n)}) = 2^{-n}$, and the Boolean algebra generated by $\bigcup_n \mathcal{P}_n$ dense in MALG_μ. Denote by $\tilde{G}_n$ the subgroup of $\tilde{G}$ consisting of all functions which are constant on each piece of the partition $\mathcal{P}_n$. It is then clear that the groups G^{2^n} and $\tilde{G}_n$ are isometrically (with respect to $d_{2^n}, \tilde{d}|\tilde{G}_n$) isomorphic, so we can identify G^{2^n} and $\tilde{G}_n$ and thus $G^{(\infty)}$ with $\tilde{G}_{(\infty)} = \bigcup_n \tilde{G}_n$.

Proposition 19.5. *$\tilde{G}_\infty$ is dense in $\tilde{G}$ and thus $\tilde{G}$ is separable.*

Proof. Using the separability of G, it is easy to check that the $f \in \tilde{G}$ with countable range are dense in $\tilde{G}$, from which it immediately follows that the $f \in \tilde{G}$ with finite range are dense in $\tilde{G}$. Given such an f with range $\{g_1, \dots, g_k\}$, let $A_i = f^{-1}(\{g_i\})$, and for each $\epsilon > 0$ find n large enough so that there is a partition $B_1, \dots, B_k$ consisting of elements of the Boolean algebra generated by $\mathcal{P}_n$, with $\mu(A_i \Delta B_i) < \epsilon/k, \forall i \leq k$. If $g \in \tilde{G}$ is defined by $g(x) = g_i$, for $x \in B_i$, then $g \in \tilde{G}_n$ and $\tilde{d}(f, g) < \epsilon$. □

Finally we have:

Proposition 19.6. *If the metric d on G is complete, so is the metric $\tilde{d}$ on $\tilde{G}$. Thus $\tilde{G}$ is a Polish group.*

Proof. Let $\{f_n\}$ be $\tilde{d}$-Cauchy, i.e., $\int d(f_n(x), f_m(x)) d\mu(x) \to 0$. Fix $\epsilon > 0$. Then there is N such that for $m, n \geq N$, $\int d(f_m(x), f_n(x)) d\mu(x) < \epsilon^2$, so $\mu(\{x : d(f_m(x), f_n(x))\} > \epsilon\} < \epsilon$. Thus we can find $n_1 < n_2 < \dots$ such that if $A_i = \{x : d(f_{n_i}(x), f_{n_{i+1}}(x)) > 2^{-i}\}$, then $\mu(A_i) < 2^{-i}$. Thus, by Borel-Cantelli, $\exists N \forall i \geq N (x \notin A_i)$, a.e.$(x)$, i.e., $\exists N \forall i \geq N (d(f_{n_i}(x), f_{n_{i+1}}(x)) \leq 2^{-i})$, a.e.$(x)$, so $\{f_{n_i}(x)\}$ is d-Cauchy and $f_{n_i}(x) \to f(x)$, a.e., for some $f \in \tilde{G}$. Using Lebesgue Dominated Convergence, $\tilde{d}(f_{n_i}, f) \to 0$ and so $\tilde{d}(f_n, f) \to 0$. □

Thus for each Polish group G, we have a canonical way of enlarging G to a Polish group $\tilde{G}$ in which G is a closed subgroup. An interesting property of this extension is the following.

Proposition 19.7. *For any Polish group G, $\tilde{G}$ is contractible.*

Proof. We can take $(X,\mu) = ([0,1],\lambda)$. For $f \in \tilde{G}, t \in [0,1]$ let $f_t(x) = f(x)$, if $x > t$; $= 1$, if $x \le t$. Thus $f_0 = f, f_1 = 1$. Also if $s < t, \tilde{d}(f_s, g_t) \le (t-s) + \tilde{d}(f,g)$, so $(f,t) \mapsto f_t$ is continuous and a contraction of $\tilde{G}$ to 1. □

We next note that we can extend the definition of $L(X,\mu,G)$ to the case of a σ-finite measure. So let X be a standard Borel space, ν a non-atomic σ-finite measure on X and let $\mu \sim \nu$ be an equivalent probability measure. Then we define $L(X,\nu,G)$ to be $L(X,\mu,G)$ (which clearly only depends on ν). The topology $\tau_{X,\mu,G}$ depends only on the measure class of μ, thus only on ν, and we can denote it by $\tau_{X,\nu,G}$. We also have the compatible metric $d_{X,\mu,G}$, which of course depends on μ.

Fix now a complete metric d on G. Consider the complete metric group $(\tilde{G},\tilde{d}) = (L(X,\mu,G), d_{X,\mu,G})$. Then we have a canonical shift action of $\mathrm{Aut}(X,\mu)$ by isometric isomorphisms of $(\tilde{G},\tilde{d})$, which is an analog of the Koopman representation of $\mathrm{Aut}(X,\mu)$ on $L^2(X,\mu)$. Namely, for any $T \in \mathrm{Aut}(X,\mu), f \in L(X,\mu,G)$, let $T \cdot f(x) = f(T^{-1}(x))$. Thus to each $T \in \mathrm{Aut}(X,\mu)$ we associate the isometric isomorphism

$$\iota_T(f) = f \circ T^{-1}$$

on $(\tilde{G},\tilde{d})$. Endowing, as usual, the isometry group $\mathrm{Iso}(\tilde{G},\tilde{d})$ with the pointwise convergence topology, under which it is a Polish group, we have the following fact.

Proposition 19.8. *If G is non-trivial, the map $T \mapsto \iota_T$ is a topological group isomorphism of* $(\mathrm{Aut}(X,\mu), w)$ *with a (necessarily closed) subgroup of* $\mathrm{Iso}(\tilde{G},\tilde{d})$. *In particular, the action $T \cdot f$ of* $\mathrm{Aut}(X,\mu)$ *on $L(X,\mu,G)$ is continuous.*

Proof. This map is clearly a homomorphism from $\mathrm{Aut}(X,\mu)$ into the group $\mathrm{Iso}(\tilde{G},\tilde{d})$. If $\iota_T = 1$, then $f(T^{-1}(x)) = f(x)$, a.e., $\forall f \in \tilde{G}$. Fix $g_0 \neq 1, g_0 \in G$. Let for $A \subseteq X, \chi_A(x) = g_0$, if $x \in A$; $= 1$, if $x \notin A$. If A is Borel, then $\chi_A(T^{-1}(x)) = \chi_{T(A)}(x)$, a.e, so $T(A) = A$. Thus $T = 1$.

Clearly $T \mapsto \iota_T$ is Borel, thus continuous. Conversely, if $\iota_{T_n} \to 1$, then for any Borel $A, \iota_{T_n}(\chi_A) \to \chi_A$ (in $\tilde{G}$) or $\int d(\chi_{T_n(A)}(x), \chi_A(x))d\mu(x) = \int \chi_{T_n(A)\Delta A}(x) d(1,g_0) d\mu(x) = \mu(T_n(A)\Delta A) d(1,g_0) \to 0$, i.e., $T_n \to 1$. □

Remark. It is clear that if G is abelian, so is $\tilde{G}$. Glasner [Gl1] has shown that $\tilde{\mathbb{T}}$ is monothetic.

Remark. Pestov [Pe1] has shown that if G is amenable locally compact, then $\tilde{G}$ is extremely amenable.

Remark. Let $G = \mathbb{Z}_2$ and d be the discrete metric on G. Then clearly $\tilde{G}$ can be identified with $(\mathrm{MALG}_\mu, \Delta)$ and $\tilde{d}(A,B) = d_\mu(A,B) = \mu(A\Delta B)$. In this case, via $T \mapsto \iota_T$ defined above, $\mathrm{Aut}(X,\mu)$ is identified with the group of all isometries S of $(\mathrm{MALG}_\mu, d_\mu)$ such that $S(\emptyset) = \emptyset$.

Since $\mathrm{Aut}(X, \mu)$ acts on $\tilde{G}$ by shift, $T \cdot f = f \circ T^{-1}$, we can form the semidirect product

$$\mathrm{Aut}(X, \mu) \ltimes \tilde{G},$$

which is the product space $\mathrm{Aut}(X, \mu) \times \tilde{G}$ with multiplication

$$(S, f)(T, g) = (ST, (T^{-1} \cdot f)g).$$

See Appendix I for more about semidirect products and, in particular, see part **(B)** for our choice of the multiplication rule.

Clearly $\mathrm{Aut}(X, \mu) \ltimes \tilde{G}$ is a Polish group (when $\mathrm{Aut}(X, \mu)$ is equipped with the weak topology) and the canonical action of $\mathrm{Aut}(X, \mu)$ on the group $\mathrm{Aut}(X, \mu) \ltimes \tilde{G}$ given by

$$T \cdot (S, f) = (TST^{-1}, T \cdot f)$$

is a continuous action by automorphisms.

Comments. The basic facts about the group $L(X, \mu, G)$ can be found in Moore [Mo], which lays the foundation of Moore cohomology theory (note that one can also consider the analogous metric space $L(X, \mu, M)$ for any given metric space M). The earlier paper Hartman-Mycielski [HM] gives a version of this construction with emphasis on the connectedness properties of the resulting groups.

20. Cocycles

(A) Let $a \in A(\Gamma, X, \mu)$ be a measure preserving action of a countable group Γ on (X, μ) and let G be a Polish group. A *cocycle* of a (with values in G) is a Borel map $\alpha : \Gamma \times X \to G$ such that for all $\gamma, \delta \in \Gamma$, writing $a(\gamma, x) = \gamma \cdot x$, we have the *cocycle identity*:

$$(*) \qquad \alpha(\gamma\delta, x) = \alpha(\gamma, \delta \cdot x)\alpha(\delta, x), \ \mu\text{-a.e.}(x).$$

We identify two cocycles α, β if for all $\gamma, \alpha(\gamma, x) = \beta(\gamma, x)$, a.e.$(x)$. Thus we could equivalently consider measurable instead of Borel cocycles. Note that the cocycle identity implies $\alpha(1, x) = 1, \alpha(\gamma, x)^{-1} = \alpha(\gamma^{-1}, \gamma \cdot x)$, a.e.$(x)$.

We denote by

$$Z^1(a, G)$$

the set of cocycles for the action a. Thus $Z^1(a, G) \subseteq L(\Gamma \times X, \eta_\Gamma \times \mu, G)$, where η_Γ is the *counting measure* on Γ.

When Γ acts in a Borel way on a standard Borel space X a *strict cocycle* for this action is a Borel map $\alpha : \Gamma \times X \to G$ such that $\alpha(\gamma\delta, x) = \alpha(\gamma, \delta \cdot x)\alpha(\delta, x)$ for all $\gamma, \delta \in \Gamma$, and *all* $x \in X$. It is easy to check that given $\alpha \in Z^1(a, G)$, there is a strict cocycle α' such that for all $\gamma, \alpha(\gamma, x) = \alpha'(\gamma, x)$, a.e.$(x)$. Indeed, there is a Borel set $A \subseteq X$ of measure 1 which is invariant under the action a and $(*)$ holds for all $\gamma, \delta \in \Gamma, x \in A$. Then put $\alpha'(\gamma, x) = \alpha(\gamma, x)$, if $x \in A$, and $\alpha'(\gamma, x) = 1$, if $x \notin A$.

When $\alpha \in Z^1(a, G)$ is independent of x, i.e., $\alpha(\gamma, x) = \varphi(\gamma)$,, where $\varphi : \Gamma \to G$, then (*) implies that φ is a homomorphism. Thus the homomorphisms $\varphi : \Gamma \to G$ are exactly the cocycles that depend only on $\gamma \in \Gamma$.

The group $L(X, \mu, G) = \tilde{G}$ acts on $Z^1(a, G)$ by

$$f \cdot \alpha(\gamma, x) = f(\gamma \cdot x)\alpha(\gamma, x)f(x)^{-1}.$$

We say that α, β are *cohomologous cocycles* or *equivalent cocycles*, in symbols

$$\alpha \sim \beta,$$

if there is $f \in \tilde{G}$ with $f \cdot \alpha = \beta$. We denote by

$$[\alpha]_\sim$$

the *cohomology class* of α. The *trivial cocycle* is denoted by 1: $\alpha(\gamma, x) = 1$. A cocycle α is a *coboundary* if it is cohomologous to 1, i.e., for some $f \in \tilde{G}$

$$\alpha(\gamma, x) = f(\gamma \cdot x)f(x)^{-1}.$$

The set of coboundaries is denoted by

$$B^1(a, G).$$

Finally the quotient space

$$H^1(a, G) = Z^1(a, G)/ \sim$$

is called the *(1st-)cohomology space* of a, relative to G.

In the special case where G is abelian, $Z^1(a, G)$ is an abelian group under pointwise multiplication, $B^1(a, G)$ is a subgroup and clearly $\sim$ is induced by the cosets of $B^1(a, G)$, so

$$H^1(a, G) = Z^1(a, G)/B^1(a, G)$$

is an abelian group, the *(1st-)cohomology group* of a.

If we view an action $a \in A(\Gamma, X, \mu)$ as a homomorphism

$$a \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(X, \mu)),$$

where (by abusing notation) $a(\gamma) = \gamma^a = (x \mapsto a(\gamma, x))$ and a cocycle $\alpha : \Gamma \times X \to G$ as a map $\alpha : \Gamma \to L(X, \mu, G)$, where $\alpha(\gamma)(x) = \alpha(\gamma, x)$, then, as explained in Appendix I, the pair (a, α) gives an extension of the homomorphism $a \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(X, \mu))$ to a homomorphism $\gamma \mapsto (a(\gamma), \alpha(\gamma))$ of Γ into $\mathrm{Aut}(X, \mu) \ltimes \tilde{G}$, where this semidirect product is defined as in Section 19, **(B)**. Thus if we put

$$AZ^1(\Gamma, X, \mu, G) = \{(a, \alpha) : a \in A(\Gamma, X, \mu), \alpha \in Z^1(a, G)\},$$

then we can identify $AZ^1(\Gamma, X, \mu, G)$ with $\mathrm{Hom}(\Gamma, \mathrm{Aut}(X, \mu) \ltimes \tilde{G})$.

Recall that $\mathrm{Aut}(X, \mu)$ acts on $\mathrm{Aut}(X, \mu) \ltimes \tilde{G}$ by

$$T \cdot (S, f) = (TST^{-1}, T \cdot f),$$

where $T \cdot f = f \circ T^{-1}$, and thus acts on $AZ^1(\Gamma, X, \mu, G)$ by

$$T \cdot (a, \alpha) = (TaT^{-1}, T \cdot \alpha).$$

where $T \cdot \alpha(\gamma, x) = \alpha(\gamma, T^{-1}(x))$.

Let C_a be the *stabilizer* of the action a, i.e.,

$$C_a = \{T : TaT^{-1} = a\}.$$

This is a closed subgroup of $\mathrm{Aut}(X,\mu)$. Then C_a acts on $Z_1(a,G)$ via $(T,\alpha) \mapsto T\cdot\alpha$. Note that this action preserves $\sim$, i.e.,

$$\alpha \sim \beta \Rightarrow T\cdot\alpha \sim T\cdot\beta$$

and so preserves $B^1(a,G)$. It also induces an action on $H^1(a,G)$.

Consider now the semidirect product $C_a \ltimes \tilde{G}$ (a closed subgroup of the group $\mathrm{Aut}(X,\mu) \ltimes \tilde{G}$). It acts on $Z^1(a,G)$ by

$$(T,g)\cdot\alpha(\gamma,x) = g(\gamma\cdot T^{-1}(x))\alpha(\gamma,T^{-1}(x))g(T^{-1}(x))^{-1}.$$

If α,β belong to the same orbit of this action then we say that α,β are *weakly equivalent*, in symbols,

$$\alpha \sim^w \beta.$$

We can easily see that

$$\alpha \sim^w \beta \leftrightarrow \exists T \in C_a(T\cdot\alpha \sim \beta).$$

More generically, if $a_i \in A(\Gamma, X_i, \mu_i), i = 1,2, \alpha_i \in Z^1(a_i,G)$, we say that α_1,α_2 are *weakly equivalent*, $\alpha_1 \sim^w \alpha_2$, if there is an isomorphism of the actions a_1,a_2 that sends α_1 to a cocycle equivalent to α_2.

(B) Let now E be a countable, measure preserving equivalence relation on (X,μ). Recall from Section 6, **(B)** that we can define a σ-finite Borel measure M on $E \subseteq X^2$ by $M(A) = \int \mathrm{card}(A_x)d\mu(x) = \int \mathrm{card}(A^y)d\mu(y)$, for Borel $A \subseteq E$. Note that $A \subseteq E$ is M-conull iff there is an E-invariant Borel set $B \subseteq X$ of measure 1 such that for $x,y \in B, xEy \Rightarrow (x,y) \in A$.

A *cocycle* of E with values in G is a Borel map $\alpha : E \to G$ which satisfies the *cocycle identity*

$$\alpha(x,z) = \alpha(y,z)\alpha(x,y),$$

for all $xEyEz$ in some E-invariant Borel subset of X of measure 1. We identify two cocycles α,β if $\alpha(x,y) = \beta(x,y), M-$a.e.$(x,y)$. Again instead of Borel we could equivalently consider $(M-)$measurable cocycles. If α is a cocycle, then $\alpha(x,x) = 1$ and $\alpha(x,y) = \alpha(y,x)^{-1}$, a.e.

We denote by

$$Z^1(E,G)$$

the set of cocycles for E with values in G, so that $Z^1(E,G) \subseteq L(E,M,G)$.

Again for any countable Borel equivalence relation E on a standard Borel space X, a *strict cocycle* is a Borel map $\alpha : E \to G$ such that $\alpha(x,z) = \alpha(y,z)\alpha(x,y)$ for *all* $xEyEz$. It is easy to see that for any $\alpha \in Z^1(E,G)$ there is a strict cocycle α' such that $\alpha(x,y) = \alpha'(x,y),\ M-$a.e.$(x,y)$.

The group $L(X,\mu,G) = \tilde{G}$ acts on $Z^1(E,G)$ by

$$f\cdot\alpha(x,y) = f(y)\alpha(x,y)f(x)^{-1},$$

and we say that α, β are *cohomologous cocycles* or *equivalent cocycles*, in symbols

$$\alpha \sim \beta,$$

if they belong to the same orbit of this action and we denote by $[\alpha]_\sim$ the *cohomology class* of α. The cohomology class of the trivial cocycle $\alpha(x,y) = 1$ is the set of *coboundaries*, denoted by

$$B^1(E,G),$$

and the quotient space

$$H^1(E,G) = Z^1(E,G)/\sim$$

is the *(1st-)cohomology space* of E, relative to G. Again if G is abelian, $B^1(E,G) \leq Z^1(E,G)$ and $H^1(E,G) = Z^1(E,G)/B^1(E,G)$ are also abelian groups, the latter called the *(1st-)cohomology group* of E.

The automorphism group $N[E]$ of $[E]$ acts on $Z^1(E,G)$ via

$$T \cdot \alpha(x,y) = \alpha(T^{-1}(x), T^{-1}(y)),$$

and this action preserves $\sim$ and $B^1(E,G)$ and thus descends to an action on $H^1(E,G)$. Finally $N[E] \ltimes L(X,\mu,G) \leq \mathrm{Aut}(X,\mu) \ltimes L(X,\mu,G)$ acts on $Z^1(E,G)$ by

$$(T,g) \cdot \alpha(x,y) = g(T^{-1}(y))\alpha(T^{-1}(x), T^{-1}(y))g(T^{-1}(x))^{-1}$$

and induces the notion of weak equivalence,

$$\alpha \sim^w \beta,$$

among cocycles. Again

$$\alpha \sim^w \beta \Leftrightarrow \exists T \in N[E](T \cdot \alpha \sim \beta).$$

Similarly we define the notion of weak equivalence of $\alpha_i \in Z^1(E_i, G), i = 1, 2$.

Remark. Note that if $T \in [E]$, then $T \cdot \alpha \sim \alpha$. Because if $f(x) = \alpha(x, T^{-1}(x))$, then

$$\begin{aligned} f \cdot \alpha(x,y) = f(y)\alpha(x,y)f(x)^{-1} &= \alpha(y, T^{-1}(y))\alpha(x,y)\alpha(T^{-1}(x), x) \\ &= \alpha(T^{-1}(x), T^{-1}(y)) = T \cdot \alpha(x,y), \end{aligned}$$

so $\alpha \sim f \cdot \alpha = T \cdot \alpha$.

(C) Suppose now that $a \in A(\Gamma, X, \mu)$ and let E_a be the associated equivalence relation. Then there is a canonical embedding of $Z^1(E_a, G)$ into $Z^1(a, G)$ given by

$$\alpha(x,y) \mapsto \alpha^*(\gamma, x) = \alpha(x, \gamma \cdot x).$$

(To see that this is 1-1 note that if $\alpha \neq \beta$, so that $M(\{(x,y) \in E_a : \alpha(x,y) \neq \beta(x,y)\}) > 0$, then for some $\gamma \in \Gamma, M(\{(x, \gamma \cdot x) : \alpha(x, \gamma \cdot x) \neq \beta(x, \gamma \cdot x)\}) > 0$, so $\mu(\{x : \alpha^*(\gamma, x) \neq \beta^*(\gamma, x)\}) > 0$.) Clearly $\alpha \mapsto \alpha^*$ preserves the action of $L(X,\mu,G)$ on $Z^1(E_a, G), Z^1(a, G)$, so sends $Z^1(E_a, G)$ onto a cohomology invariant subset of $Z^1(a, G)$. Also it maps $B^1(E_a, G)$ onto $B^1(a, G)$.

We note that $\alpha \mapsto \alpha^*$ maps $Z^1(E_a, G)$ *onto* $Z^1(a, G)$ iff a is free. Indeed, if a is free, and $\alpha \in Z^1(a, G)$, let $\beta(x, y) = \alpha(\gamma, x)$, where $\gamma \cdot x = y$. Then $\beta^* = \alpha$. Conversely, if a is not free let $\beta(\gamma, x) = \gamma$. If $\alpha^* = \beta$, then $\gamma = \alpha(x, \gamma \cdot x)$. Let x be such that $\gamma \cdot x = x$ for some $\gamma \neq 1$. Then $\gamma = \alpha(x, \gamma \cdot x) = \alpha(x, x) = 1$, a contradiction.

Remark. Suppose that a is not necessarily free but E_a is treeable with associated treeing $\mathcal{T} \subseteq E_a$. Let $\Gamma = \{\gamma_n\}$ and find Borel $e : \mathcal{T} \to \Gamma$ with $e(x, y) \cdot x = y$ and $e(y, x) = e(x, y)^{-1}$. Also let $e(x, x) = 1$. Extend e to $\epsilon : E \to \Gamma$ by $\epsilon(x, y) = e(x_n, y) \cdots e(x_1, x_2) e(x, x_1)$, where $x, x_1, \ldots, x_n, y$ is the unique $\mathcal{T}$-path from x to y. Clearly $\epsilon \in Z^1(E_a, \Gamma)$ and $\epsilon(x, y) \cdot x = y$. Define now

$$\beta \in Z^1(a, G) \mapsto \beta_* \in Z^1(E_a, G)$$

by

$$\beta_*(x, y) = \beta(\epsilon(x, y), x).$$

Then $(\alpha^*)_* = \alpha$, so $\beta \mapsto \beta_*$ is a map from $Z^1(a, G)$ onto $Z^1(E_a, G)$, which is the inverse of $\alpha \mapsto \alpha^*$ on $Z^1(E_a, G)^*$.

(D) Suppose that $\{\gamma_i\}_{i \in I}$ (finite or infinite) is a set of generators for Γ. Fix $a \in A(\Gamma, X, \mu)$. Given $\alpha \in Z^1(a, G)$, put $f_{\alpha,i}(x) = \alpha(\gamma_i, x)$. Clearly $f_{\alpha,i} \in L(X, \mu, G) = \tilde{G}$. Since for $\delta, \delta_1, \ldots, \delta_n \in \Gamma$,

$$\alpha(\delta_1 \delta_2 \cdots \delta_n, x) = \alpha(\delta_1, \delta_2 \cdots \delta_n \cdot x) \cdots \alpha(\delta_{n-1}, \delta_n \cdot x) \alpha(\delta_n, x),$$

and $\alpha(\delta^{-1}, x) = \alpha(\delta, \delta^{-1} \cdot x)^{-1}$, it follows that the maps $\{f_{\alpha,i}\}_{i \in I}$ completely determine α. When $\Gamma = F_n$ and $\gamma_1, \ldots, \gamma_n$ are free generators, then $\alpha \mapsto (f_{\alpha,1}, \ldots, f_{\alpha,n})$ is a 1-1 correspondence between $Z^1(a, G)$ and $(\tilde{G})^n$, so $Z^1(a, G)$ can be identified with $(\tilde{G})^n$. When G is abelian, this is an identification of the group $Z^1(a, G)$ with the product group $(\tilde{G})^n$. Thus in the particular case $\Gamma = F_1 = \mathbb{Z}, Z^1(a, G)$ can be identified with $L(X, \mu, G)$ with $\alpha \in Z^1(a, G)$ identified with $f(x) = \alpha(1, x)$ (then note that $\alpha(n, x) = f((n-1) \cdot x) \ldots f(1 \cdot x) f(x)$ if $n > 0$ and $\alpha(n, x) = \alpha(-n, n \cdot x)^{-1}$, if $n < 0$).

(E) Given two homomorphisms $\varphi, \psi : \Gamma \to G$ between countable groups Γ, G viewed as cocycles in $Z^1(a, G)$ for a given $a \in A(\Gamma, X, \mu)$, we can determine when φ, ψ are cohomologous. Call φ, ψ *conjugate* if there is $g \in G$ such that $g\varphi(\gamma)g^{-1} = \psi(\gamma), \forall \gamma \in \Gamma$.

Proposition 20.1. *Let Γ be a countable group, G a countable group and $\varphi, \psi \in \mathrm{Hom}(\Gamma, G)$. If $a \in A(\Gamma, X, \mu)$ is such that every finite index subgroup of Γ acts ergodically (e.g., if a is weak mixing), then (viewing φ, ψ as cocycles in $Z^1(a, G)$),*

$$\varphi \sim \psi \Leftrightarrow \varphi, \psi \textit{ are conjugate.}$$

Proof. $\Leftarrow$ is obvious.

$\Rightarrow$: Assume that for some $f \in \tilde{G}$,

$$\psi(\gamma) = f(\gamma \cdot x) \varphi(\gamma) f(x)^{-1}.$$

Let $g_0 \in G$ be such that $f^{-1}(\{g_0\}) = A_0$ has positive measure. Then

$$x, \gamma \cdot x \in A_0 \Rightarrow \psi(\gamma) = g_0 \varphi(\gamma) g_0^{-1}.$$

Put $\Gamma_0 = \{\gamma \in \Gamma : \psi(\gamma) = g_0 \varphi(\gamma) g_0^{-1}\}$. Then $\Gamma_0 \leq \Gamma$ and $x, \gamma \cdot x \in A_0 \Rightarrow \gamma \in \Gamma_0$. Conversely, if $\gamma \in \Gamma_0, x \in A_0$, then $\psi(\gamma) = f(\gamma \cdot x)\varphi(\gamma) g_0^{-1}$, so $g_0 \varphi(\gamma) g_0^{-1} = f(\gamma \cdot x)\varphi(\gamma) g_0^{-1}$, thus $f(\gamma \cdot x) = g_0$ and so $\gamma \cdot x \in A_0$. It follows that A_0 is Γ_0-invariant and A_0 meets every Γ-orbit in a single Γ_0-orbit (modulo null sets).

Let $\{\gamma_i\}_{i \in I}$ be a set of left-coset representatives for Γ_0. Then we claim that $i \neq j \Rightarrow \gamma_i \cdot A_0 \cap \gamma_j \cdot A_0 = \emptyset$. Otherwise, there are $x_0, y_0 \in A_0$ with $\gamma_i \cdot x_0 = \gamma_j \cdot y_0$, so x_0, y_0 are in the same Γ-orbit, thus the same Γ_0-orbit. Let $\gamma_0 \cdot x_0 = y_0$, for some $\gamma_0 \in \Gamma_0$. Then $\gamma_0^{-1} \gamma_j^{-1} \gamma_i \cdot x_0 = x_0$, so $\gamma_0^{-1} \gamma_j^{-1} \gamma_i \in \Gamma_0$, i.e., $\gamma_i \in \gamma_j \Gamma_0$, a contradiction.

Therefore I is finite and thus $[\Gamma : \Gamma_0] < \infty$. So Γ_0 acts ergodically and therefore $\mu(A_0) = 1$ and φ, ψ are conjugate. □

In particular, under the hypotheses of the proposition, the only homomorphism in $B^1(a, G)$ is the trivial one.

Remark. The condition that every finite index subgroup of Γ acts ergodically is necessary. Let $\Gamma = \Gamma_0 \times \mathbb{Z}_2$, where Γ_0 is an infinite group. Let $a \in A(\Gamma_0, X, \mu)$ be free and ergodic, let $Y = X \times \{0\} \cup X \times \{1\} = X_0 \cup X_1$ be the union of two disjoint copies of X, and let $\nu = \frac{1}{2}(\mu_0 + \mu_1)$, with μ_i the copy of μ in X_i. Let Γ_0 act on Y in the obvious way: $\gamma \cdot (x, i) = (\gamma \cdot x, i)$. Let $\mathbb{Z}_2$ act on Y by the involution $T(x, i) = (x, 1-i)$. This action commutes with the Γ_0-action, so it gives an action of Γ which is free and ergodic. However, it is obviously not Γ_0-ergodic. Let $\varphi : \Gamma \to \mathbb{Z}_2$ be the projection map. Then $\varphi \sim 1$ (since $\varphi(\gamma) = f(\gamma \cdot y) - f(y)$, where $f(x, i) = i$) but $\varphi \neq 1$.

(F) Let $E_1 \subseteq E_2$ be two equivalence relations on (X, μ). Then there is a canonical restriction map

$$\alpha \in Z^1(E_2, G) \mapsto \alpha | E_1 \in Z^1(E_1, G).$$

It clearly respects the action of $\tilde{G}$ and thus $\alpha \sim \beta \Rightarrow \alpha | E_1 \sim \beta | E_1$.

(G) Consider next an equivalence relation E on (X, μ) and let $A \subseteq X$ be a Borel complete section a.e. (i.e., A meets almost all E-classes). Then $E|A$ is an equivalence relation on (A, μ_A), where $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$ is the normalized restriction of μ to A. For each $\alpha \in Z^1(E, G)$, we denote by $\alpha | A$ its restriction to A, so that $\alpha | A \in Z^1(E|A, G)$. Conversely, fix a Borel map $f_A : X \to A$ such that $f_A(x) E x$, a.e., and $f_A | A = \mathrm{id}$. Then given $\beta \in Z^1(E|A, G)$, define $\beta^A \in Z^1(E, G)$, by $\beta^A(x, y) = \beta(f_A(x), f_A(y))$. Thus $\beta^A | A = \beta$. It is clear that

$$\beta_1 \sim \beta_2 \Leftrightarrow \beta_1^A \sim \beta_2^A.$$

We also claim that

$$\alpha_1 \sim \alpha_2 \Leftrightarrow \alpha_1 | A \sim \alpha_2 | A.$$

The direction $\Rightarrow$ is obvious. Conversely, assume that there is $g : A \to G$ such that for $uEv, u, v \in A$, we have

$$\alpha_2(u,v) = g(v)\alpha_1(u,v)g(u)^{-1}.$$

Let then $f : X \to G$ be defined by

$$f(x) = \alpha_2(f_A(x), x)g(f_A(x))\alpha_1(x, f_A(x)).$$

Then for xEy we have

$$\begin{aligned}
& f(y)\alpha_1(x,y)f(x)^{-1} \\
= \; & \alpha_2(f_A(y), y)g(f_A(y))\alpha_1(y, f_A(y))\alpha_1(x,y) \\
& \alpha_1(f_A(x), x)g(f_A(x))^{-1}\alpha_2(x, f_A(x)) \\
= \; & \alpha_2(f_A(y), y)g(f_A(y))\alpha_1(f_A(x), f_A(y))g(f_A(x))^{-1}\alpha_2(x, f_A(x)) \\
= \; & \alpha_2(f_A(y), y)\alpha_2(f_A(x), f_A(y))\alpha_2(x, f_A(x)) \\
= \; & \alpha_2(x,y).
\end{aligned}$$

In particular, for any $\alpha \in Z^1(E,G)$

$$\alpha \sim (\alpha|A)^A$$

(since $\alpha|A = (\alpha|A)^A|A$). Thus the map

$$\beta \in Z^1(E|A, G) \mapsto \beta^A \in Z^1(E,G)$$

is a bijection from $Z^1(E|A,G)$ onto a subset $Z^1_A \subseteq Z^1(E,G)$, which is a complete section of $\sim$ on $Z^1(E,G)$. So in a sense the cohomology spaces $H^1(E,G)$ and $H^1(E|A,G)$ are isomorphic.

Comments. Some basic references for the theory of cocycles are the following: Feldman-Moore [FM], Schmidt [Sc1], [Sc5] and Zimmer [Zi].

21. The Mackey action and reduction cocycles

(A) We will work for a while in a pure Borel theoretic context with no measures present.

Let E be a countable Borel equivalence relation on a standard Borel space X, let G be a Polish group, and $\alpha : E \to G$ a strict Borel cocycle. Consider the space $X \times G$ and the equivalence relation R_α on $X \times G$ defined by

$$(x,g)R_\alpha(y,h) \Leftrightarrow xEy \ \& \ \alpha(x,y)g = h.$$

Let

$$X_\alpha = (X \times G)/R_\alpha$$

be the quotient space, which is not necessarily a standard Borel space. The group G acts in a Borel way on $X \times G$ by

$$g \cdot (x,h) = (x, hg^{-1})$$

and this action respects R_α, so G acts on X_α by

$$g \cdot [x,h] = [x, hg^{-1}],$$

where $[x, h] = [(x, h)]_{R_\alpha}$. Note that this action is free.

Recall that when a group H acts on a set Y, we denote by E_H^Y the equivalence relation induced by this action. Let $p_\alpha : X \to X_\alpha$ be defined by

$$p_\alpha(x) = [x, 1].$$

Then it can be easily seen that p_α is a *reduction* of E to $E_G^{X_\alpha}$, i.e.,

$$xEy \Leftrightarrow p_\alpha(x) E_G^{X_\alpha} p_\alpha(y),$$

and α is the *associated cocycle* to p_α in the sense that

$$xEy \Rightarrow \alpha(x, y) \cdot p_\alpha(x) = p_\alpha(y).$$

Moreover the reduction p_α admits a Borel lifting, i.e., there is a Borel map $\varphi_\alpha : X \to X \times G$ (namely $\varphi_\alpha(x) = (x, 1)$) with $p_\alpha(x) = [\varphi_\alpha(x)]$.

Conversely, let Y be a standard Borel space, F a Borel equivalence relation on Y, G a Polish group and assume G acts freely on Y/F so that this action has a Borel lifting, in the sense that there is a Borel map $\psi : G \times Y \to Y$ such that $g \cdot [y]_F = [\psi(g, y)]_F$. If $p : X \to Y/F$ is a reduction of E to $E_G^{Y/F}$, i.e., $xEy \Leftrightarrow p(x) E_G^{Y/F} p(y)$, and p admits a Borel lifting, i.e., there is Borel $\varphi : X \to Y$ with $[\varphi(x)]_F = p(x)$, then we can define the (strict) Borel cocycle $\alpha : E \to G$ associated to this reduction by $\alpha(x, y) \cdot p(x) = p(y)$ (to see that α is Borel notice that $\alpha(x, y) = g \Leftrightarrow g \cdot [\varphi(x)]_F = [\varphi(y)]_F \Leftrightarrow \psi(g, \varphi(x)) F \varphi(y)$, so the graph of α is Borel).

Let us note now that the map $p_\alpha : X \to X_\alpha$ has range $p_\alpha(X)$ which is a complete section for $E_G^{X_\alpha}$, i.e., $G \cdot p_\alpha(X) = X_\alpha$. We now have the following uniqueness property.

Proposition 21.1. *Let E be a countable Borel equivalence relation, G a Polish group and $\alpha : E \to G$ a strict Borel cocycle. Let Y be a standard Borel space, F a Borel equivalence relation on Y and assume G acts freely on Y/F with Borel lifting and $p : X \to Y/F$ is a reduction of E to $E_G^{Y/F}$ with Borel lifting, such that $G \cdot p(X) = Y/F$ and p has associated cocycle α. Then there is a unique isomorphism $\pi : X_\alpha \to Y/F$ of G-actions such that $\pi \circ p_\alpha = p$. Moreover, π has a Borel lifting (i.e., there is Borel $\theta : X \times G \to Y$ with $\pi([x, g]) = [\theta(x, g)]_F$).*

Proof. To prove uniqueness, assume that π_1, π_2 are two such maps. Then $\pi_1([x, 1]) = p(x) = \pi_2([x, 1])$, so $\pi_1([x, g]) = \pi_1(g^{-1} \cdot [x, 1]) = g^{-1} \cdot \pi_1([x, 1]) = g^{-1} \cdot \pi_2([x, 1]) = \pi_2([x, g])$.

Now put

$$\pi([x, g]) = g^{-1} \cdot p(x).$$

It is easy to check that this is well-defined and works. □

Let now $\alpha \sim \beta$ in the Borel sense, i.e., there is Borel $f : X \to G$ with

$$f(y)\alpha(x, y)f(x)^{-1} = \beta(x, y).$$

Let α come from a reduction p of E to $E_G^{Y/F}$ as in 21.1. Then let

$$q(x) = f(x) \cdot p(x),$$

$q : X \to Y/F$. Then q is a reduction with Borel lifting of E to $E_G^{Y/F}$ with associated cocycle $\beta(x, y)$. Moreover $q(x)E_G^{Y/F}p(x)$. Clearly $(X_\alpha, p_\alpha; Y/F, p)$, $(X_\beta, p_\beta; Y/F, q)$ are canonically isomorphic by 21.1 and chasing diagrams this shows that X_α, X_β are isomorphic G-spaces via the isomorphism

$$\rho : X_\alpha \to X_\beta$$

given by

$$\rho([x, g]_{R_\alpha}) = [x, f(x)g]_{R_\beta}$$

and moreover $\rho(p_\alpha(x))E_G^{X_\beta}p_\beta(x)$. Conversely, if $\rho : X_\alpha \to X_\beta$ is an isomorphism of G-actions with Borel lifting and $\rho(p_\alpha(x))E_G^{X_\beta}p_\beta(x)$ with $\rho(p_\alpha(x)) = f(x)^{-1} \cdot p_\beta(x)$, then f is Borel and

$$f(y)\alpha(x, y)f(x)^{-1} = \beta(x, y),$$

so $\alpha \sim \beta$, in the Borel sense.

Thus there is a canonical 1-1 correspondence between $f : X \to G$ as above, witnessing that $\alpha \sim \beta$, in the Borel sense, and isomorphisms of G-actions $\rho : X_\alpha \to X_\beta$ with Borel liftings for which

$$\rho(p_\alpha(x))E_G^{X_\beta}p_\beta(x).$$

The action of G on the space X_α is called the *Mackey action* (or *Mackey range* or *Poincaré flow*) associated with the cocycle α. By the above it is an invariant of cohomology (in the pure Borel context).

If a countable group Γ acts in a Borel way on X and $\alpha : \Gamma \times X \to G$ is a (strict) Borel cocycle for this action, then one can define the skew product action of Γ on $X \times G$ given by

$$\gamma \cdot (x, g) = (\gamma \cdot x, \alpha(\gamma, x)g).$$

Denoting by R_α the induced equivalence relation, we can define the Mackey action as before and prove similar facts.

In particular, if E is a countable Borel equivalence relation induced by a Borel action of a countable group Γ on $X, E = E_\Gamma^X$, and $\alpha : E \to G$ is a Borel cocycle, then $\alpha^*(\gamma, x) = \alpha(x, \gamma \cdot x)$ is a Borel cocycle for the action and it is easy to check that $R_{\alpha^*} = R_\alpha$, so R_α is induced by the skew product action of Γ on $X \times G$ associated to α^*.

(B) We are of course primarily interested in the case where the space X_α is standard, i.e., where the countable Borel equivalence relation R_α is smooth (equivalently admits a Borel selector). Let us introduce the following notions.

For each Borel cocycle $\alpha : E \to G$ the *kernel* of α is the equivalence relation

$$xE_\alpha y \Leftrightarrow xEy \ \& \ \alpha(x, y) = 1.$$

We say that α is *smooth* if E_α is smooth.

A Borel cocycle $\alpha : E \to G$ is called a *reduction cocycle* if there is a free Borel action of G on a *standard Borel space* Y and a Borel reduction $p : X \to Y$ of E to E_G^Y with associated cocycle $\alpha, \alpha(x,y) \cdot p(x) = p(y)$.

More generally, if G, Y are as before and $p : X \to Y$ is only a *Borel homomorphism* of E to E_G^Y, i.e.,

$$xEy \Rightarrow p(x) E_G^Y p(y),$$

then p has an associated cocycle $\alpha : E \to G$ given by $\alpha(x,y) \cdot p(x) = p(y)$. Again if $\alpha \sim \beta$ with $\beta(x,y) = f(y)\alpha(x,y)f(x)^{-1}$, where $f : X \to G$ is Borel, then letting $q(x) = f(x) \cdot p(x)$, we have that q is a Borel homomorphism of E to E_G^Y such that $q(x) E_G^Y p(x)$ and q has associated cocycle β.

We now have the following fact:

Proposition 21.2. *Let E be a countable Borel equivalence relation, G a countable group and $\alpha : E \to G$ a Borel cocycle. Then the following are equivalent:*

(i) α is smooth,

(ii) R_α is smooth,

(iii) α is a reduction cocycle,

(iv) α is associated to a countable-to-1 Borel homomorphism.

Proof. The previous analysis of the Mackey action shows that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) is clear as if α is associated to the reduction p, then $xE_\alpha y \Leftrightarrow p(x) = p(y)$. Clearly (iii) $\Rightarrow$ (iv) and we can see that (iv) $\Rightarrow$ (i) as follows: If p is a countable-to-1 homomorphism and $xEy \Rightarrow \alpha(x,y) \cdot p(x) = p(y)$, then $E_\alpha \subseteq \{(x,y) : p(x) = p(y)\} = F$, which is a smooth countable Borel equivalence relation, thus E_α is smooth.

Finally, we verify that (i) $\Rightarrow$ (ii): Let $c = [x,g]_{R_\alpha}$. Let $g_c =$ the least g' such that $\exists y (y, g') \in c$ ("least" means in some fixed enumeration of G). Then if $(y, g_c), (z, g_c) \in c$, we have $yE_\alpha z$. There is Borel $f : X \times G \to X \times G$ such that $f(x,g) = (y, g_c)$, where $c = [x,g]_{R_\alpha}, (y, g_c) \in c$. Then $(x,g) R_\alpha (y,h) \Leftrightarrow f(x,g) E_\alpha \times \Delta(G) f(y,h)$, where $\Delta(G)$ is the equality relation on G. Thus R_α is smooth. □

A special case of a reduction cocycle comes from isomorphism. Let G be a countable group. A Borel cocycle $\alpha : E \to G$ is called an *isomorphism cocycle* if there is a free Borel action of G on a standard Borel space Y and a Borel isomorphism $p : X \to Y$ of E with E_G^Y with associated cocycle $\alpha, \alpha(x,y) \cdot p(x) = p(y)$. In this case α has the following property:

(*) For each $x \in X, y \mapsto \alpha(x,y)$ is a bijection of $[x]_E$ with G.

Conversely, if α satisfies (*), then it is easy to check that it is an isomorphism cocycle. Indeed, notice that (*) implies that $\alpha(x,y) = 1 \Leftrightarrow x = y$, so α is smooth and thus the space X_α of the Mackey action of G is standard Borel, and of course the action is free. Finally (*) implies that the canonical map $p_\alpha(x) = [x, 1]$ is a Borel isomorphism of E with $E_G^{X_\alpha}$.

(C) We now move on to the measure theoretic framework. Let E be a countable Borel, measure preserving equivalence relation on $(X, \mu), G$ a locally compact Polish group, η_G a left-invariant Haar measure, and $\mu_G \sim \eta_G$ an equivalent probability measure on G. Given $\alpha \in Z^1(E, G)$, which we can assume to be strict, consider the product space $(X \times G, \mu \times \mu_G)$ and the equivalence relation R_α given as before by

$$(x, g) R_\alpha (y, h) \Leftrightarrow xEy \ \& \ \alpha(x, y) g = h.$$

It is easy to see that R_α is $(\mu \times \eta_G)-$measure preserving and therefore it is $(\mu \times \mu_G)-$quasi-invariant, i.e., R_α-saturations of null sets are null.

Now let $\mathcal{S}_\alpha$ be the σ-subalgebra of $\mathrm{MALG}_{\mu \times \mu_G}$ which consists of the R_α-invariant sets. Then there is (an essentially unique) standard Borel space X_α and a Borel map $\pi_\alpha : X \times G \to X_\alpha$, which is R_α-invariant, and such that if $(\pi_\alpha)_*(\mu \times \mu_G) = \mu_\alpha$, then $A \mapsto (\pi_\alpha)^{-1}(A)$ is an isomorphism of $\mathrm{MALG}_{\mu_\alpha}$ with $\mathcal{S}_\alpha$ (see, e.g., Kechris [Kec2], 17.43).

Now G acts on $X \times G$ by

$$g \cdot (x, h) = (x, hg^{-1})$$

and $\mu \times \mu_G$ is quasi-invariant under this action, which also preserves R_α and thus $\mathcal{S}_\alpha$. Then by a result of Mackey [M] (see also Kechris [Kec1], 3.47), there is (an essentially unique) Borel action of G on X_α such that μ_α is quasi-invariant and $\pi_\alpha : X \times G \to X_\alpha$ preserves the G-actions. The G-action on (X_α, μ_α) is called the *Mackey action* (or *Mackey range* or *Poincaré flow*) associated to α (in the measure theoretic context).

Define now $p_\alpha : X \to X_\alpha$ by

$$p_\alpha(x) = \pi_\alpha(x, 1).$$

Then $xEy \Rightarrow p_\alpha(x) E_G^{X_\alpha} p_\alpha(y) \ \& \ \alpha(x, y) \cdot p_\alpha(x) = p_\alpha(y)$. However the action of G on X_α might not be free (in fact X_α can be trivial), and so p_α might not be a reduction.

If $\alpha \sim \beta$ and $f : X \to G$ is such that $f(y)\alpha(x, y)f(x)^{-1} = \beta(x, y)$, let $F : X \times G \to X \times G$ be defined by

$$F(x, g) = (x, f(x)g).$$

Then F is a Borel isomorphism of R_α with R_β and also of the G-action and moreover preserves $\mu \times \eta_G$. It therefore induces an isomorphism of the Mackey actions of α, β. Thus the Mackey action is an invariant of cohomology.

One can similarly define the Mackey action associated to an action $a \in A(\Gamma, X, \mu)$ and cocycle $\alpha \in Z^1(a, G)$.

We say that $\alpha \in Z^1(E, G)$ is a *reduction cocycle* if its restriction to an E-invariant Borel set of measure 1 is a reduction cocycle (in the previous Borel sense). Similarly, we define what it means for α to be associated to a Borel homomorphism. Again if α is a reduction cocycle (resp., associated to a homomorphism) and $\beta \sim \alpha$, then β is a reduction cocycle (resp., associated to a homomorphism). Similarly, we say that α is *smooth* if its restriction to an E-invariant Borel set of measure 1 is smooth (in the Borel sense). Also R_α is smooth if its restriction to a co-null R_α-invariant Borel set is smooth (in the Borel sense).

Finally, we recall the following notion of transience for cocycles. The cocycle $\alpha \in Z^1(E, G)$ is *transient* if there is a set $A \subseteq X$ of positive measure and an open nbhd V of 1 such that if $x, y \in A, xEy, x \neq y$, then $\alpha(x, y) \notin V$. Otherwise α is called *recurrent.*

We now have the following analog of 21.2.

Proposition 21.3. *The following are equivalent for $\alpha \in Z^1(E, G), E$ ergodic and G countable:*

(i) α is smooth,
(ii) R_α is smooth,
(iii) α is a reduction cocycle,
(iv) α is associated to a countable-to-1 Borel homomorphism,
(v) α is transient.

Proof. The equivalence of (i)-(iv) comes from 21.2.

(i) $\Rightarrow$ (v): Let $A \subseteq X$ be a Borel transversal for E_α. Then $\mu(A) > 0$ and if $x, y \in A, xEy, x \neq y$, then $\alpha(x, y) \neq 1$.

(v) $\Rightarrow$ (ii): Fix A witnessing that α is transient. Then $A \times \{1\} \subseteq X \times G$ is a partial transversal for R_α (i.e., meets every R_α class in at most one point) and has positive $\mu \times \eta_G$-measure. Call an R_α-invariant Borel set $B \subseteq X \times G$ *smooth* if $R_\alpha | B$ is smooth. By the countable chain condition there is a largest (modulo null sets) smooth set B_0. As $A \times \{1\} \subseteq B_0$, clearly $(\mu \times \eta_G)(B_0) > 0$. Since the action of G preserves R_α, B_0 is G-invariant, so has the form $B_0 = A_0 \times G$, for some $A_0 \subseteq X$ of positive measure. Moreover A_0 is E-invariant (as $x \in A_0, yEx \Rightarrow (x, g) R_\alpha (y, \alpha(x, y)g)$, so $(y, \alpha(x, y)g) \in B_0$ and $y \in A_0$). By ergodicity of E, $\mu(A_0) = 1$, so $B_0 = X \times G$, modulo null sets, and R_α is smooth. □

Remark. Note that (in the context of 21.3) no $\alpha \in B^1(E, G)$ is transient, otherwise α would be a reduction cocycle, so $1 \in B^1(E, G)$ would be a reduction cocycle, thus E would be smooth, a contradiction.

In the measure theoretic context, we say that $\alpha \in Z^1(E, G), G$ a countable group, is an *isomorphism cocycle* if there is a free action of G on a standard Borel space Y and a Borel isomorphism π of E restricted to an invariant conull Borel set with E_G^Y such that $\alpha(x, y) \cdot \pi(x) = \pi(y)$. Such cocycles can again be characterized by the condition that $y \mapsto \alpha(x, y)$ is a bijection between $[x]_E$ and G, a.e.(x). Similarly, if $a \in \mathrm{FR}(\Gamma, X, \mu), b \in \mathrm{FR}(G, Y, \nu)$, then an orbit equivalence π between a, b gives rise to a cocycle α as above, which we call an *orbit equivalence cocycle.*

Because cocycles arise canonically from reductions of equivalence relations and, in particular, from orbit equivalences, the study of cocycles has numerous applications in the study of orbit equivalence in ergodic theory as well as in the descriptive theory of Borel equivalence relations; see, for example, Zimmer [Zi2], Adams-Kechris [AK], Popa [Po2], for a sample of such applications.

When E is ergodic, then the Mackey action on (X_α, μ_α) is also ergodic, since a G-invariant subset of X_α corresponds to a subset of $X \times G$ which is both R_α-invariant and G-invariant and thus either null or co-null.

On the other hand the ergodicity of R_α is only true for certain cocycles α.

Given a cocycle $\alpha \in Z^1(E, G)$, we define its *essential range* $ER(\alpha) \subseteq G$ by

$$ER(\alpha) = \{g \in G : \forall \text{ open nbhd } V \text{ of } g\ \forall A \subseteq X(\mu(A) > 0 \Rightarrow \exists x, y \in A, xEy(\alpha(x, y) \in V))\}.$$

Note that if $g \in ER(\alpha)$, then for every open nbhd V of g and any $A \subseteq X$ of positive measure, $\{x \in A : \exists y(y \in A \ \&\ xEy \ \&\ \alpha(x, y) \in V)\}$ has actually the same measure as A. It is not hard to see that $ER(\alpha)$ is a closed subgroup of G. First it is clear that $1 \in ER(\alpha)$ and $ER(\alpha)$ is closed under inverses. Let now $g, h \in ER(\alpha)$. Fix a countable group Γ and a Borel action of Γ on X with $E^X_\Gamma = E$. Fix an open nbhd W of gh and open nbhds U, V of g, h with $UV \subseteq W$. Let $\mu(A) > 0$. Then we can find $\gamma \in \Gamma$, such that if $C = \{x \in A : \gamma \cdot x \in A \ \&\ \alpha(x, \gamma \cdot x) \in U\}$, then $\mu(C) > 0$ and of course $C \cup \gamma \cdot C \subseteq A$. Similarly there is $\delta \in \Gamma, D \subseteq C$ with $\mu(D) > 0$, $D \cup \delta \cdot D \subseteq C$ and $\alpha(x, \delta \cdot x) \in V, \forall x \in D$. Then if $x \in D, y = \gamma\delta \cdot x \in A$ and if $z = \delta \cdot x (\in C \subseteq A)$, then $\alpha(x, y) = \alpha(z, y)\alpha(x, z) = \alpha(z, \gamma \cdot z)\alpha(x, \delta \cdot x) \in UV \subseteq W$. So $gh \in ER(\alpha)$. That $ER(\alpha)$ is closed is obvious.

When G is countable, which is the case we are mostly interested in here, $ER(\alpha)$ is the subgroup of G consisting of all $g \in G$ that are in the range of α restricted to every set of positive measure.

We now have the following well-known characterization.

Proposition 21.4. *Let E be ergodic, G countable and $\alpha \in Z^1(E, G)$. Then R_α is ergodic (i.e., the Mackey action is trivial) iff $ER(\alpha) = G$.*

Proof. Assume first that R_α is ergodic and fix $A \subseteq X$ with $\mu(A) > 0$ and $g \in G$. Since $A \times \{g\}, A \times \{1\}$ have positive measure, they meet almost every R_α-class, so there are $x \in A, y \in A$ such that $(x, 1)R_\alpha(y, g)$. Then xEy and $\alpha(x, y) = g$

Conversely, assume that $ER(\alpha) = G$. We will show that the R_α-saturation of any positive measure set is conull. For this it is enough to show that if $A \subseteq X, \mu(A) > 0$ and $g_0 \in G$, then for almost all x and any $h_0 \in G$ we have $(x, h_0)R_\alpha(y, g_0)$ for some $y \in A$. Recall that for each $g \in G, \{x \in A : \exists y \in A(xEy \ \&\ \alpha(x, y) = g)\}$ has the same measure as A, so $B = \{x : \forall g \exists y \in A(xEy \ \&\ \alpha(x, y) = g)\}$ has positive measure and since it is E-invariant, it has measure 1. If $x \in B$, then there is $y \in A$ with xEy and $\alpha(x, y) = g_0 h_0^{-1}$, so $\alpha(x, y)h_0 = g_0$ and $(x, h_0)R_\alpha(y, g_0)$. □

In general, the conjugacy class of $ER(\alpha)$ is not a cohomology invariant (see, e.g., Arnold-Nguyen Dinh Cong-Oseledets [ANO]) but the normal subgroup

$$NER(\alpha) = \bigcap_{g \in G} (gER(\alpha)g^{-1})$$

is a cohomology invariant. Indeed, let $\alpha \sim \beta$ and assume that $ghg^{-1} \in ER(\alpha), \forall g$, in order to show that $h \in ER(\beta)$. Fix f such that $\beta(x,y) = f(y)\alpha(x,y)f(x)^{-1}$. Let W be an open set containing h, and find open sets U, V with $h \in U, 1 \in V$ such that $VV^{-1}UVV^{-1} \subseteq W$. Let $\{g_i\}$ be dense in G, so that $\bigcup_i Vg_i = G$. Let $\mu(A) > 0$. Fix then i such that $B = A \cap \{x : f(x) \in Vg_i\}$ has positive measure. Since $g_i^{-1}hg_i \in ER(\alpha)$, we can find $x, y \in B$ such that $\alpha(x,y) \in g_i^{-1}V^{-1}hVg_i$ and so $\beta(x,y) \in Vg_ig_i^{-1}V^{-1}hVg_ig_i^{-1}V^{-1} = VV^{-1}hVV^{-1} \subseteq W$.

If we denote, for Polish locally compact G, by $\overline{G} = G \cup \{\infty\}$ the one-point compactification of G, then we let for any cocycle $\alpha \in Z^1(E,G)$,

$$\overline{ER}(\alpha) = \{\overline{g} \in \overline{G} : \forall \text{ open nbhd } V \text{ of } \overline{g}\ \forall A \subseteq X(\mu(A) > 0 \Rightarrow$$

$$\exists x, y \in A, xEy(\alpha(x,y) \in V))\}.$$

Thus $ER(\alpha) = \overline{ER}(\alpha) \cap G$. We note now the following fact (see Schmidt [Sc1], 3.15).

Proposition 21.5. *Let E be ergodic, G countable and $\alpha \in Z^1(E,G)$. If α is transient, then $\overline{ER}(\alpha) = \{1, \infty\}$. In particular, $ER(\alpha) = \{1\}$.*

Proof. Clearly $1 \in \overline{ER}(\alpha)$. Fix $A \subseteq X$ with $\mu(A) > 0$ and finite $K \subseteq G$. Then $E_\alpha|A$ is smooth, so let T be a Borel transversal for it. Since $E|A$ is ergodic, each $E|A$-class contains infinitely many $E_\alpha|A$-classes. Fix an $E|A$-class C and let $T \cap C = \{x_1, x_2, \dots\}$, where $x_i \neq x_j$ if $i \neq j$. We claim that for some $i, \alpha(x_1, x_i) \notin K$, which shows that $\infty \in \overline{ER}(\alpha)$. Otherwise, if $K = \{g_1, \dots, g_n\}$, for some $k \leq n$ and infinitely many i, $\alpha(x_1, x_i) = g_k$, so there are $i \neq j$ with $\alpha(x_i, x_j) = 1$, i.e., $x_i E_\alpha x_j$, a contradiction.

Finally, we show that $ER(\alpha) = \{1\}$. Otherwise let $g \in ER(\alpha), g \neq 1$. We will show that for every set $A \subseteq X$ of positive measure there are $x \neq y \in A, xEy$ and $\alpha(x,y) = 1$, which implies that α is recurrent, a contradiction.

Let Γ be a countable group acting in a Borel way on X so that $E = E_\Gamma^X$. Then there is some γ such that $B = \{x \in A : \gamma \cdot x \in A \ \&\ \alpha(x, \gamma \cdot x) = g\}$ has positive measure. Note that if $x \in B$, then $\alpha(x, \gamma \cdot x) \neq 1$, so $\gamma \cdot x \neq x$. Thus we can find Borel $C \subseteq B$ of positive measure such that $C \cup \gamma \cdot C \subseteq A, C \cap \gamma \cdot C = \emptyset$ and $\alpha(x, \gamma \cdot x) = g, \forall x \in C$. Since $g^{-1} \in ER(\alpha)$, we can find $\delta \in \Gamma, D \subseteq \gamma \cdot C$ of positive measure such that $D \cup \delta \cdot D \subseteq \gamma \cdot C$ and $\alpha(y, \delta \cdot y) = g^{-1}, \forall y \in D$. Then if $x \in \gamma^{-1} \cdot D \subseteq C$ $(\subseteq A), y = \gamma \cdot x \in D, \delta \cdot y = \delta\gamma \cdot x \in A$, $x \neq \delta\gamma \cdot x$ and $\alpha(x, \delta\gamma \cdot x) = \alpha(\gamma \cdot x, \delta\gamma \cdot x)\alpha(x, \gamma \cdot x) = \alpha(y, \delta \cdot y)\alpha(x, \gamma \cdot x) = g^{-1}g = 1$. □

Next we give some examples of cocycles α with $ER(\alpha) = G$.

Let H be a countable dense subgroup of a compact group K. Let $\Gamma = H \oplus H \oplus \dots$. Let $\varphi : \Gamma \to G$ be an epimorphism such that for each $n, \varphi|\{1\} \oplus \dots \oplus \{1\} \oplus H \oplus H \oplus \dots$, where there are n 1's, is also an epimorphism.

Consider the compact group $K^{\mathbb{N}}$ and the dense subgroup $\Gamma \leq K^{\mathbb{N}}$. We let Γ act by left-translation on $(K^{\mathbb{N}}, \mu)$, where μ is the Haar measure on $K^{\mathbb{N}}$, so that this action is free, measure-preserving, and ergodic. Let $E = E_{\Gamma}^{K^{\mathbb{N}}}$. We will define an $\alpha \in Z^1(E, G)$ with $ER(\alpha) = G$. If xEy, and $\gamma \in \Gamma$ is such that $\gamma \cdot x = \gamma x = y$, put $\alpha(x, y) = \varphi(\gamma)$. (Thus if we view α as a cocycle of the Γ-action, α is a homomorphism into G.) We will show that $ER(\alpha) = G$. Fix $g \in G$ and let $A \subseteq K^{\mathbb{N}}$ have positive measure. Then AA^{-1} contains a non-empty nbhd of 1, so for some n, $\{1\} \oplus \cdots \oplus \{1\} \oplus H \oplus H \oplus \cdots \subseteq AA^{-1}$. Let $(h_1, h_2, \dots) \in H \oplus H \oplus \dots$ be such that $\varphi(1, 1, \dots 1, h_1, h_2, \dots) = g$. If $\gamma = (1, 1, \dots, 1, h_1, h_2, \dots)$, then there are $x, y \in A$ with $\gamma = yx^{-1}$, so $y = \gamma x = \gamma \cdot x$ and $\alpha(x, y) = \varphi(\gamma) = g$.

For instance, we can take $H = \mathbb{Z}$, $K = \mathbb{T}$ and G any countable abelian group. Clearly there is $\varphi : \Gamma \to G$ with the above property. Then E as above is an ergodic hyperfinite equivalence relation and $\alpha \in Z^1(E, G)$ has $ER(\alpha) = G$.

(D) We have seen earlier that examples of cocycles in $Z^1(E, G)$ arise in a natural way from homomorphisms of the equivalence relation E to equivalence relations induced by free Borel actions of G on a standard Borel space. It is reasonable to ask whether there are cocycles that do not arise in this fashion. We now give a simple example of this sort.

Let Γ be a non-amenable countable group that admits an epimorphism $\varphi : \Gamma \to G$ onto a non-trivial abelian group G with infinite kernel. Let $a \in A(\Gamma, X, \mu)$ be free, mixing and E_0-ergodic and let $E = E_a$. We will show that there is a cocycle $\alpha \in Z^1(E, G)$ such that α is not associated to any Borel homomorphism $p : X \to Y$ of E to E_G^Y, where G acts in a Borel way freely on Y. First note that the cocycle $\alpha(x, y) = \varphi(\gamma)$, where $\gamma \cdot x = y$, is not a coboundary. Otherwise $\alpha(x, y) = f(y)f(x)^{-1}$, for some $f : X \to G$, so that on a set of positive measure $A \subseteq X$, $f|A$ is constant. Then $x, y \in A, \gamma \cdot x = y \Rightarrow \gamma \in \ker(\varphi) = \Gamma_0$. So the intersection of $\Gamma_0 \cdot A$ with any Γ-orbit is a single Γ_0-orbit. Since the action is Γ_0-ergodic, $\Gamma_0 \cdot A = X$ and thus $\Gamma = \Gamma_0$, a contradiction. We now argue that α cannot be associated to a p, E_G^Y as above. Otherwise, $xEy \Rightarrow p(y) = \alpha(x, y) \cdot p(x)$ and since E is E_0-ergodic, and G is abelian, there is $y_0 \in Y$ such that $p(x) E_G^Y y_0$, a.e.(x). So $p(x) = f(x) \cdot y_0$, for some Borel $f : X \to G$. Then $\alpha(x, y) = f(y)f(x)^{-1}$, a contradiction.

22. Homogeneous spaces and Effros' Theorem

(A) Before we continue with the theory of cocycles, in this and the next section we will discuss some general facts concerning continuous actions, that we will need later on. We review first some basic results about homogeneous spaces and discuss Effros' Theorem.

Let G be a Polish group and $H \leq G$ a closed subgroup. We endow G/H with the quotient topology, under which it is a Polish space. If d is a right-invariant compatible metric on G, then

$$\begin{aligned} d(xH, yH) &= \inf\{d(u,v) : u \in xH, v \in yH\} \\ &= \inf\{d(x,v) : v \in yH\} \end{aligned}$$

is a compatible metric for G/H. The group G acts on G/H by $g \cdot xH = gxH$ and this action is continuous.

If d is (two-sided) invariant, then G acts on G/H by isometries. Moreover d on G/H is complete. To see this, suppose $d(x_nH, x_mH) \to 0$. Then find $n_1 < n_2 < \dots$ with $d(x_{n_i}H, x_{n_{i+1}}H) = d(x_{n_i}, x_{n_{i+1}}H) < 2^{-i}$. We can then find $h_1, h_2, \dots \in H$ with $d(x_{n_1}, x_{n_2}h_1) < 1/2, d(x_{n_2}, x_{n_3}h_2) < 1/4, \dots$, so $d(x_{n_1}, x_{n_2}h_1) < 1/2, d(x_{n_2}h_1, x_{n_3}h_2h_1) < 1/4, \dots$ and the sequence

$$\{x_{n_1}, x_{n_2}h_1, x_{n_3}h_2h_1, \dots\}$$

is d-Cauchy in G. Since d is complete (being invariant), there is $f \in G$ with $x_{n_i}h_{i-1}h_{i-2}\dots h_1 \to f$, so $d(f, x_{n_i}H) \to 0$ and thus $d(x_{n_i}H, fH) \to 0$, i.e., $x_{n_i}H \to fH$ in G/H and so $\{x_nH\}$ converges.

(B) We recall *Effros' Theorem* (see Effros [Ef2], Kechris [Kec1] and Becker-Kechris [BK]).

Theorem 22.1 (Effros). *Let G be a Polish group acting continuously on a Polish space P. For each $p \in P$, let $G_p = \{g \in G : g \cdot p = p\}$ be the stabilizer of p.*

(a) The following are equivalent for any $p \in P$:

(i) The canonical map $gG_p \mapsto g \cdot p$ is a homeomorphism of G/G_p onto $G \cdot p$ (equivalently $g \mapsto g \cdot p$ is open from G onto $G \cdot p$),

(ii) $G \cdot p$ is not meager in its relative topology,

(iii) $G \cdot p$ is G_δ in P.

(b) If moreover E_G^P is F_σ (in P^2), then E_G^P is smooth iff every G-orbit is locally closed (i.e., the difference of two closed sets or equivalently open in its closure).

Corollary 22.2 (Marker, Sami). *In the context of 22.1 (a), if $G \cdot p$ is nonmeager in P, then $G \cdot p$ is G_δ.*

Below we use the following notation:

$$\begin{aligned} &\forall^\infty n R(n) \Leftrightarrow \exists N \forall n \geq N R(n) \\ &\exists^\infty n R(n) \Leftrightarrow \forall n \exists n \geq N R(n). \end{aligned}$$

Corollary 22.3. *If a countable group G acts by homeomorphisms on a Polish space P, then the following are equivalent:*

(i) E_G^P is smooth,

(ii) $\forall p \in P \forall \{g_n\} \in G^{\mathbb{N}}[g_n \cdot p \to p \Rightarrow \forall^\infty n(g_n \cdot p = p)]$,

(iii) $\forall p \in P \forall \{g_n\} \in G^{\mathbb{N}}(g_n \cdot p \to p \Rightarrow \exists n(g_n \cdot p = p)]$.

Proof. The equivalence of (ii), (iii) is clear. Now we have, since E_G^P is F_σ and G is countable:

$$\begin{aligned} E_G^P \text{ is smooth } &\Leftrightarrow \forall p (G \cdot p \text{ is locally closed}) \\ &\Leftrightarrow \forall p (p \text{ is isolated in } \overline{G \cdot p}) \\ &\Leftrightarrow \forall p (p \text{ is isolated in } G \cdot p) \\ &\Leftrightarrow \forall p \forall \{g_n\} \in G^{\mathbb{N}} (g_n \cdot p \to p \Rightarrow \ \forall^\infty n (g_n \cdot p = p)\}. \end{aligned}$$

□

Since a point $p \in P$ is called *recurrent* if for every open $V \ni p$, there is $g \in G$ with $p \neq g \cdot p \in V$, 22.3 says that E_G^P is not smooth iff there is a recurrent point.

23. Isometric actions

We will discuss here some general results concerning isometric actions.

Suppose G is a Polish group, (M, ρ) a Polish metric space (i.e., ρ is a complete separable metric) and $(g, x) \mapsto g \cdot x$ a continuous action of G by isometries on (M, ρ). As with any continuous action of a Polish group on a Polish space, we consider the topological ergodic decomposition of this action induced by the G_δ equivalence relation.

$$x \approx y \Leftrightarrow \overline{G \cdot x} = \overline{G \cdot y}$$

and the corresponding G_δ partial ordering

$$x \preccurlyeq y \Leftrightarrow x \in \overline{G \cdot y} \Leftrightarrow \overline{G \cdot x} \subseteq \overline{G \cdot y}.$$

Proposition 23.1. *The partial order $x \preccurlyeq y$ is an equivalence relation and thus the equivalence classes of the topological ergodic decomposition are closed and consist of the orbit closures.*

Proof. Assume $x \preccurlyeq y$ and let $g_n \in G$ be such that $g_n \cdot y \to x$. Then $\rho(g_n \cdot y, x) \to 0$, so $\rho(y, g_n^{-1} \cdot x) \to 0$ and $y \preccurlyeq x$. □

Recall that for $A, B \subseteq M$,

$$\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B).$$

Proposition 23.2. *If $\overline{G \cdot x} \neq \overline{G \cdot y}$, then $\rho(\overline{G \cdot x}, \overline{G \cdot y}) > 0$.*

Proof. Otherwise, let g_n, h_n be such that $\rho(g_n \cdot x, h_n \cdot y) \to 0$, so $\rho(h_n^{-1} g_n \cdot x, y) \to 0$, thus $y \preccurlyeq x$, and so $y \approx x$, a contradiction. □

Proposition 23.3. *$\approx$ is a closed equivalence relation.*

Proof. Let $\{V_n\}$ be an open basis for M. Put $W_n = G \cdot V_n$ and

$$(x, y) \in R \Leftrightarrow \exists m \exists n (x \in W_m, y \in W_n \text{ and } W_m \cap W_n = \emptyset).$$

Clearly R is open in M^2 and we prove that

$$(\approx) = (\sim R).$$

Clearly if $(x, y) \in R$, then $x \approx y$ fails. Conversely, if $\overline{G \cdot x} \neq \overline{G \cdot y}$, then $\rho(\overline{G \cdot x}, \overline{G \cdot y}) > \epsilon > 0$. So there are m, n with $x \in V_m \subseteq B_\rho(x, \epsilon/2), y \in V_n \subseteq B_\rho(y, \epsilon/2)$. If $z \in W_m \cap W_n$, then there are g, h with $g^{-1} \cdot z \in V_m, h^{-1} \cdot z \in V_n$, so $\rho(x, g^{-1} \cdot z) < \epsilon/2, \rho(y, h^{-1} \cdot z) < \epsilon/2$, thus $\rho(g \cdot x, h \cdot y) < \epsilon$, a contradiction. $\square$

Proposition 23.4. *The quotient space $M/ \approx$ (with the quotient topology) is Polish with compatible complete metric $\rho(\overline{G \cdot x}, \overline{G \cdot y})$.*

Proof. Since $\rho(\overline{G \cdot x}, \overline{G \cdot y}) = \rho(G \cdot x, G \cdot y) = \rho(x, G \cdot y)$, it follows that it satisfies the triangle inequality, and so 23.2 shows that it is a metric. We next check that it is compatible with the quotient topology.

We first verify that for $y \in M, \epsilon > 0, \{\overline{G \cdot x} : \rho(\overline{G \cdot x}, \overline{G \cdot y}) < \epsilon\}$ is open in the quotient topology or equivalently that $A = \{x : \rho(G \cdot x, G \cdot y) < \epsilon\}$ is open in M. But $x \in A \Leftrightarrow \exists g, h \in G \rho(g \cdot x, h \cdot y) < \epsilon$, so this is clear. Conversely, suppose $V \subseteq (M/ \approx)$ is open and $\overline{G \cdot x} \in V$. We will then find $\epsilon > 0$ such that $\rho(\overline{G \cdot x}, \overline{G \cdot y}) < \epsilon \Rightarrow \overline{G \cdot y} \in V$. Put $\tilde{V} = \{y : \overline{G \cdot y} \in V\}$. Then $\tilde{V}$ is open in M, G-invariant, and $x \in \tilde{V}$. So fix $\epsilon > 0$ with $B_\rho(x, \epsilon) \subseteq \tilde{V}$. If $\rho(\overline{G \cdot x}, \overline{G \cdot y}) = \rho(x, G \cdot y) < \epsilon$, there is $h \in G$ with $\rho(x, h \cdot y) < \epsilon$, so $h \cdot y \in B_\rho(x, \epsilon) \subseteq \tilde{V}$, thus $y \in h^{-1} \cdot \tilde{V} = \tilde{V}$, and $\overline{G \cdot y} \in V$.

Finally we show that $\rho(\overline{G \cdot x}, \overline{G \cdot y})$ is complete. Let $\rho(\overline{G \cdot x_n}, \overline{G \cdot x_m}) = \rho(x_n, G \cdot x_m) \to 0$. Then we can find $n_1 < n_2 < \dots$ with $\rho(x_{n_i}, G \cdot x_{n_{i+1}}) < 2^{-i}$. Choose inductively $g_{n_i} \in G$ as follows: Let $g_{n_1} = 1$. Then let g_{n_2} be such that $\rho(x_{n_1}, g_{n_2} \cdot x_{n_2}) < 2^{-1}$. As $\rho(x_{n_2}, G \cdot x_{n_3}) < 2^{-2}$, let $h \in G$ be such that $\rho(x_{n_2}, h \cdot x_{n_3}) < 2^{-2}$, so $\rho(g_{n_2} \cdot x_{n_2}, g_{n_2} h \cdot x_{n_3}) < 2^{-2}$. Put $g_{n_3} = g_{n_2} h$, etc. Thus $\rho(g_{n_i} \cdot x_{n_i}, g_{n_{i+1}} \cdot x_{n_{i+1}}) < 2^{-i}$ and so $g_{n_i} \cdot x_{n_i} \to y$ for some y, thus $\rho(\overline{G \cdot x_{n_i}}, \overline{G \cdot y}) \to 0$, i.e., some subsequence of $\overline{G \cdot x_n}$ converges in ρ and thus so does $\overline{G \cdot x_n}$. $\square$

We will now consider the structure of the orbits $G \cdot x$.

Proposition 23.5. *Assume that the Polish group G admits an invariant metric. Then if an orbit $G \cdot x$ is G_δ, it is actually closed.*

In particular, every non-meager orbit is clopen and if there is a comeager orbit the action is transitive.

Proof. Let d be an invariant metric on G. Let $H = G_x = \{g \in G : g \cdot x = x\}$. Since $G \cdot x$ is G_δ, by Effros' Theorem the canonical map $gH \mapsto g \cdot x$ is a homeomorphism of G/H with $G \cdot x$. Recall from Section 22 that d on G/H (i.e., $d(g \cdot H, h \cdot H)$) is a complete metric on G/H invariant under the G-action on G/H.

Suppose, towards a contradiction, that $g_n \cdot x \to y$, where $y \notin G \cdot x$.

Claim. *There is g such that $g_n H \to gH$ in G/H.*

Granting this, $g_n \cdot x \to g \cdot x = y$, a contradiction.

Proof of the claim. Let $\epsilon > 0$ and let $V = B_d(H, \epsilon)$ be an open ball around H in G/H. Let $U = \{g \in G : gH \in V\}$. Then $U \cdot x$ is an open nbhd of x in $G \cdot x$, so for some $\delta > 0, B_\rho(x,\delta) \cap G \cdot x \subseteq U \cdot x$. Since $\rho(g_m \cdot x, g_n \cdot x) = \rho(g_n^{-1} g_m \cdot x, x) \to 0$, we have some N such that $g_n^{-1} g_m \cdot x \in U \cdot x$, for $m, n \geq N$. So for such $m, n, g_n^{-1} g_m H \in V$, thus $d(g_n^{-1} g_m H, H) < \epsilon$ and so $d(g_m H, g_n H) < \epsilon$. Therefore $\{g_n H\}$ is d-Cauchy in G/H, thus $g_n H \to gH$ for some g.

If $G \cdot x$ is a non-meager orbit, then, by 22.2, $G \cdot x$ is G_δ so closed, and has non-empty interior, thus is also open. □

Therefore, when G admits an invariant metric, the closures of the orbits $\overline{G \cdot x}$ come in two varieties:

(i) Those that are equal to single closed orbit $\overline{G \cdot x} = G \cdot x$.

(ii) Those for which $\overline{G \cdot x}$ contains more than one orbit, i.e, $G \cdot x$ is not closed. Then the action of G on the invariant closed set $\overline{G \cdot x}$ is minimal and every orbit contained in $\overline{G \cdot x}$ is meager in $\overline{G \cdot x}$ (by 22.2 and 23.5). In that case, clearly $E_G^M|(\overline{G \cdot x})$ is not smooth, in fact E_0 can be Borel reduced to it (see Becker-Kechris [BK], 3.4.5).

So we subdivide the space M into two parts:

$$\begin{aligned} \text{SMOOTH} &= \{x \in M : G \cdot x \text{ is closed}\}, \\ \text{ROUGH} &= \{x \in M : G \cdot x \text{ is not closed}\}. \end{aligned}$$

Then SMOOTH, ROUGH are $\approx$-invariant, i.e., are unions of orbit closures.

Proposition 23.6. *Assume that the Polish group G admits an invariant metric. Then if E_G^M is Borel,* SMOOTH *and* ROUGH *are Borel.*

Proof. Denote by d an invariant metric on G. Since E_G^M is Borel, there is a Borel map $x \mapsto \{g_x^n\}_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that $\{g_x^n\}_{n \in \mathbb{N}}$ is dense in G_x (see Becker-Kechris [BK], 7.1.2). Let $\{f_n\}$ be dense in G. Then

$$\begin{aligned} x \in \text{SMOOTH} &\Leftrightarrow (g \cdot x \mapsto gG_x) \text{ is continuous} \\ &\Leftrightarrow (g \cdot x \mapsto gG_x) \text{ is continuous at } x \\ &\Leftrightarrow \forall \epsilon \exists \delta \forall g (\rho(g \cdot x, x) < \delta \Rightarrow d(gG_x, G_x) < \epsilon) \\ &\Leftrightarrow \forall \epsilon \exists \delta \forall n (\rho(f_n \cdot x, x) < \delta \Rightarrow d(f_n G_x, G_x) < \epsilon) \\ &\Leftrightarrow \forall \epsilon \exists \delta \forall n (\rho(f_n \cdot x, x) < \delta \Rightarrow \exists k \exists \ell (d(f_n g_x^k, g_x^\ell) < \epsilon)), \end{aligned}$$

where ϵ, δ vary over positive rationals. Thus SMOOTH is Borel. □

We should finally note that under the assumptions of 23.6, the equivalence relation $E_G^M|$SMOOTH is smooth, since we have that $E_G^M|$SMOOTH $= (\approx |$SMOOTH$)$, while $E_G^M|$ROUGH is not smooth, thus E_0 Borel reduces to it.

24. Topology on the space of cocycles

(A) Let G be a Polish group, (X,μ) a standard measure space and consider the semidirect product $\mathrm{Aut}(X,\mu) \ltimes L(X,\mu,G)$ as in Section 19. Given a countable discrete group Γ, we can form the topological space $(\mathrm{Aut}(X,\mu) \times L(X,\mu,G))^\Gamma = \mathrm{Aut}(X,\mu)^\Gamma \times L(X,\mu,G)^\Gamma$ with the product topology. Then

$$AZ^1(\Gamma,X,\mu,G) = \mathrm{Hom}(\Gamma,\ \mathrm{Aut}(X,\mu) \ltimes L(X,\mu,G))$$

is a closed subspace and we equip it with the subspace topology. Thus for each $a \in A(\Gamma,X,\mu) = \mathrm{Hom}(\Gamma,\mathrm{Aut}(X,\mu))$, the space $Z^1(a,G)$ is the section of $AZ^1(\Gamma,X,\mu,G)$ corresponding to a and $Z^1(a,G)$ is thus a closed subspace of $L(X,\mu,G)^\Gamma$ and therefore a Polish space. For $\alpha_n,\alpha \in Z^1(a,G)$,

$$\begin{aligned}\alpha_n \to \alpha &\Leftrightarrow \forall\gamma \in \Gamma(\alpha_n(\gamma,x) \to \alpha(\gamma,x) \text{ in measure})\\ &\Leftrightarrow \forall\gamma \in \Gamma\forall\{n_i\}\exists\{m_i\}(\{m_i\} \text{ is a subsequence of } \{n_i\}\\ &\qquad \text{and } \alpha_{m_i}(\gamma,x) \to \alpha(\gamma,x) \text{ pointwise, a.e.})\\ &\Leftrightarrow \forall\gamma \in \Gamma\Big(\int d(\alpha_n(\gamma,x),\alpha(\gamma,x))d\mu(x) \to 0\Big),\end{aligned}$$

where $d \leq 1$ is a compatible metric for G. Note that by a simple diagonal argument, we also have

$$\begin{aligned}\alpha_n \to \alpha \Leftrightarrow &\forall\{n_i\}\exists\{m_i\}(\{m_i\} \text{ is a subsequence of } \{n_i\} \text{ and}\\ &\forall\gamma(\alpha_{m_i}(\gamma,x) \to \alpha(\gamma,x) \text{ pointwise, a.e.})).\end{aligned}$$

If $\Gamma = \{\gamma_k\}_{k=1}^\infty$, then we can define a compatible metric for $Z^1(a,G)$ by

$$\begin{aligned}\tilde{d}^1(\alpha,\beta) &= d^1_{\Gamma,X,\mu,G}(\alpha,\beta)\\ &= \sum_{k=1}^\infty 2^{-k}\int d(\alpha(\gamma_k,x),\beta(\gamma_k,x))d\mu(x).\end{aligned}$$

Equivalently we can view $Z^1(a,G)$ as a closed subspace of $L(\Gamma\times X,\eta_\Gamma\times\mu,G)$, where η_Γ is the counting measure on Γ. Since $\eta_\Gamma \sim \sum_{k=1}^\infty 2^{-k}\delta_{\gamma_k}$, where $\delta_\gamma =$ Dirac measure at $\gamma \in \Gamma$, the metric on $L(\Gamma \times X,\eta_\Gamma \times \mu,G)$ associated with d is clearly $\tilde{d}^1$.

Assume now that E is a countable measure preserving equivalence relation on (X,μ). Then $Z^1(E,G) \subseteq L(E,M,G)$, where M is the usual measure on E (see Section 6, **(B)**). Moreover, it is easy to see that $Z^1(E,G)$ is a closed subspace of $L(E,M,G)$. Indeed, assume that $\alpha_n \in Z^1(E,G)$ and $\alpha_n \to f \in L(E,M,G)$. Let $A_n \subseteq X$ be E-invariant of measure 1 such that for $xEyEz$ in $A_n, \alpha_n(x,z) = \alpha_n(y,z)\alpha_n(x,y)$. Since $\alpha_n \to f$ in $L(E,M,G)$, there is a subsequence $\{n_i\}$ such that $\alpha_{n_i}(x,y) \to f(x,y)$, M-a.e. So there is E-invariant $A \subseteq \bigcap_n A_n$, with $\mu(A) = 1$, and for xEy in A, $\alpha_{n_i}(x,y) \to f(x,y)$, so that for $xEyEz$ in $A, f(x,z) = f(y,z)f(x,y)$, i.e., $f \in Z^1(E,G)$. We equip $Z^1(E,G)$ with the relative topology, so that it is again a Polish space. If $\Gamma = \{\gamma_k\}_{k=1}^\infty$ is a countable group acting in a Borel way on X with $E^X_\Gamma = E$, then $E = \bigcup_{k=1}^\infty \text{graph}(\gamma_k)$. Let μ_k be the probability measure on graph(γ_k) which is the image of μ under $x \mapsto (x, \gamma_k \cdot x)$. Then $M \sim \sum 2^{-k}\mu_k$, so a compatible metric for $Z^1(E,G)$ is

$$\tilde{d}^1(\alpha,\beta) = \sum_{k=1}^\infty 2^{-k} \int d(\alpha(x,\gamma_k\cdot x),\beta(x,\gamma_k\cdot x))d\mu(x).$$

In particular, if $a \in A(\Gamma, X, \mu)$ and $\alpha \mapsto \alpha^*$ is the canonical embedding of $Z^1(E_a, G)$ into $Z^1(a,G)$ (see Section 20, **(C)**), then it is an isometry onto a closed subspace of $Z^1(a,G)$.

Finally, if G is a Polish abelian group, then $Z^1(a,G), Z^1(E,G)$ are Polish groups, in fact $Z^1(a,G)$ (resp., $Z^1(E,G)$) is a closed subgroup of $L(\Gamma \times X, \eta_\Gamma \times \mu, G)$ (resp., $L(E,M,G)$).

(B) Consider now the action of $L(X,\mu,G)$ on $Z^1(a,G)$ as in Section 20, **(A)**. It is easy to check that it is continuous. We also have the following, when G admits an invariant metric.

Proposition 24.1. *Let G be a Polish group which admits an invariant metric $d, \Gamma = \{\gamma_k\}_{k=1}^\infty$ a countable group, $a \in A(\Gamma,X,\mu)$ and consider the metric $\tilde{d}^1$ on $Z^1(a,G)$. Then $L(X,\mu,G)$ acts by isometries on $(Z^1(a,G),\tilde{d}^1)$. Similarly for $Z^1(E,G)$.*

Proof. A trivial calculation using the above formula for $\tilde{d}^1(\alpha,\beta)$. □

As in Section 20, **(A)** the Polish group Aut(X,μ) also acts on the space $AZ^1(\Gamma,X,\mu,G)$ by $T\cdot(a,\alpha) = (TaT^{-1}, T\cdot\alpha)$, where $T\cdot\alpha(\gamma,x) = \alpha(\gamma,T^{-1}(x))$. This action is continuous and so is the action of C_a on $Z^1(a,G)$. For *any* Polish group G and compatible metric d on G, this is easily an action by isometries on $(Z^1(a,G),\tilde{d}^1)$. Also $C_a \ltimes L(X,\mu,G)$ acts continuously on $Z^1(a,G)$ and this is an action by isometries for $\tilde{d}^1$, when d is invariant.

Consider now the action of $N[E]$ on $Z^1(E,G)$ as in Section 20, **(B)**. Let E be aperiodic and consider $N[E]$ as a Polish group with the topology discussed in Section 6. We claim that this action is also continuous. For that it is enough to show that it is separately continuous.

First let $\alpha_n \to \alpha$ in $Z^1(E,G)$ and let $T \in N[E]$, in order to show that $T\cdot\alpha_n \to T\cdot\alpha$. We have that for any $\{m_i\}$ there is a subsequence $\{n_i\}$ with $\alpha_{n_i}(x,y) \to \alpha(x,y)$, M-a.e., so $\alpha_{n_i}(T^{-1}(x),T^{-1}(y)) \to \alpha(T^{-1}(x),T^{-1}(y))$, M-a.e., since $T\times T$ preserves M.

Next assume that $T_n \to T$ in $N[E]$ and let $\alpha \in Z^1(E,G)$. We need to show that $T_n\cdot\alpha \to T\cdot\alpha$ in $Z^1(E,G)$, i.e.,

$$\alpha(T_n^{-1}(x),T_n^{-1}(y)) \to \alpha(T^{-1}(x),T^{-1}(y))$$

in the space $L(E,M,G)$. Since $T\times T$ preserves M, this is equivalent to $\alpha(T_n^{-1}T(x),T_n^{-1}T(y)) \to \alpha(x,y)$. Since $T_n^{-1}T \to 1$ in $N[E]$, it is enough to show, that if $T_n \to 1$ in $N[E]$, then $T_n\cdot\alpha \to \alpha$ in $Z^1(E,G)$. Let Γ act on X so that $E^X_\Gamma = E$. Then we need to show that for each $\gamma \in \Gamma, T_n\cdot\alpha(x,\gamma\cdot x) \to$

$\alpha(x,\gamma\cdot x)$ in $L(X,\mu,G)$, i.e., $\int d(\alpha(T_n^{-1}(x),T_n^{-1}(\gamma\cdot x)),\alpha(x,\gamma\cdot x))d\mu(x)\to 0$. If this fails, then, by going to a subsequence, we can assume that for some $\epsilon>0$ the above integral is $>3\epsilon$ for each n. By 6.2, since $T_n\to 1$ in $N[E]$, we can assume that $d_u(\gamma T_n^{-1},T_n^{-1}\gamma)<\epsilon$, i.e., $\mu(\{x:\gamma\cdot T_n^{-1}(x)\neq T_n^{-1}(\gamma\cdot x)\})<\epsilon$. Thus $\int d(\alpha(T_n^{-1}(x),\gamma\cdot T_n^{-1}(x)),\alpha(x,\gamma\cdot x))d\mu(x)>\epsilon$. Let $f(x)=\alpha(x,\gamma\cdot x)$, so that $f\in L(X,\mu,G)$. Again by 6.2 we have $T_n\to 1$ in $\mathrm{Aut}(X,\mu)$ with the weak topology and since the action of $\mathrm{Aut}(X,\mu)$ on $L(X,\mu,G)$ is continuous, we have $T_n\cdot f\to f$ in $L(X,\mu,G)$, i.e., $\int d(f(T_n^{-1}(x)),f(x))d\mu(x)\to 0$, a contradiction.

Again the action of $N[E]$ on $(Z^1(a,G),\tilde{d}^1)$ is by isometries for *any* Polish group G and compatible metric d. Finally $N[E]\ltimes L(X,\mu,G)$ acts continuously on $Z^1(E,G)$ and moreover by isometries for $\tilde{d}^1$ if d is also invariant.

Remark. As we discussed in Section 20, **(D)**, there is a canonical 1-1 correspondence between $Z^1(a,G)$, where $a\in A(F_n,X,\mu)$ and $L(X,\mu,G)^n$. It is clear that this is a homeomorphism.

Remark. Let $E_1\subseteq E_2$. Then the map $\alpha\in Z^1(E_2,G)\mapsto\alpha|E_1\in Z^1(E_1,G)$ is continuous and respects the actions of $L(X,\mu,G)$.

(C) Finally recall Section 20, **(G)**. Given an equivalence relation E on (X,μ) and a complete section $A\subseteq X$, the map $\alpha\in Z^1(E,G)\mapsto\alpha|A\in Z^1(E|A,G)$ is continuous and the map $\beta\in Z^1(E|A,G)\mapsto\beta^A\in Z^1(E,G)$ is a homeomorphism of $Z^1(E|A,G)$ with a G_δ subset Z^1_A of $Z^1(E,G)$, which is a complete section of $\sim$ on $Z^1(E,G)$. Since $\alpha_1\sim\alpha_2\Leftrightarrow\alpha_1|A\sim\alpha_2|A$ and $\beta_1\sim\beta_2\Leftrightarrow\beta_1^A\sim\beta_2^A$, the Borel complexity of $\sim$ on $Z^1(E,G)$ and $\sim$ on $Z^1(E|A,G)$ is the same and similarly the Borel complexity of each cohomology class $[\alpha]_\sim$ is the same as that of $[\alpha|A]_\sim$.

25. Cohomology I: Some general facts

We will study from now on the structure of the equivalence relation $\sim$ of cohomology on $Z^1(a,G)$ and $Z^1(E,G)$. We are primarily interested in the case where G is countable but occasionally we will formulate statements in greater generality.

If the Polish group G admits an invariant metric d, which is necessarily complete, then, as we have seen in Section 24, the Polish group $\tilde{G}=L(X,\mu,G)$, which admits the invariant metric $\tilde{d}$, acts continuously by isometries on $(Z^1(a,G),\tilde{d}^1)$ and $(Z^1(E,G),\tilde{d}^1)$. So the theory of Section 23 applies. Of course the equivalence relation induced by this action is the relation of cohomology, $\sim$. We will denote by $\mathrm{ROUGH}(a,G)$ (resp., $\mathrm{SMOOTH}(a,G)$) the rough, resp., smooth parts of this action. Similarly for $\mathrm{ROUGH}(E,G)$, $\mathrm{SMOOTH}(E,G)$.

In view of 23.6, we will consider conditions under which $\sim$ $(=E_{\tilde{G}}^{Z^1(a,G)})$ is Borel.

Proposition 25.1. *Let G be a Polish group admitting an invariant metric d. Let $a\in\mathrm{ERG}(\Gamma,X,\mu)$. Then for any $\alpha\in Z^1(a,G)$, there is a closed* *subgroup $H \leq G$ and an isometric isomorphism of the stabilizer $(\tilde{G}_\alpha, \tilde{d})$ with (H, d). Similarly for any ergodic E and $\alpha \in Z^1(E, G)$. In particular, if G is locally compact (resp., countable), so is each stabilizer $\tilde{G}_\alpha$.*

Proof. For each $f, g \in \tilde{G}_\alpha$, we have

$$f(\gamma \cdot x) = \alpha(\gamma, x) f(x) \alpha(\gamma, x)^{-1},$$
$$g(\gamma \cdot x) = \alpha(\gamma, x) g(x) \alpha(\gamma, x)^{-1}.$$

Thus $d(f(\gamma \cdot x), g(\gamma \cdot x)) = d(f(x), g(x))$ and $x \mapsto d(f(x), g(x))$ is Γ-invariant, so, by ergodicity, constant a.e. Thus

$$d(f(x), g(x)) = \tilde{d}(f, g), \text{ a.e.}(x).$$

Let now $G_0 \subseteq \tilde{G}_\alpha$ be a dense subgroup of $\tilde{G}_\alpha$. Then

$$\forall f, g \in G_0 [d(f(x), g(x)) = \tilde{d}(f, g)], \text{ a.e.}(x).$$

For each $g \in G_0$, recall that g is an equivalence class of Borel functions from X to G modulo null sets. Pick a representative $\overline{g} : X \to G$ for each $g \in G_0$. Then for almost all x, and all $f, g \in G_0$ we have

$$d(\overline{f}(x), \overline{g}(x)) = \tilde{d}(f, g)$$

and $\overline{1}(x) = 1, \overline{fg}(x) = \overline{f}(x)\overline{g}(x), \overline{(f^{-1})}(x) = (\overline{f})^{-1}(x)$. So fix $x_0 \in X$ such that all of the above hold and let $F : G_0 \to G$ be defined by

$$F(g) = \overline{g}(x_0).$$

Then clearly $F : (G_0, \tilde{d}) \to (G, d)$ is an isometric group embedding and thus uniquely extends to an isometric isomorphism of $\tilde{G}_\alpha$ onto a closed subgroup H of G. □

Corollary 25.2 (Hjorth [Hj1]). *In the context of 25.1, if G is additionally locally compact (e.g., if G is countable), then the cohomology relation $\sim$ is Borel.*

Proof. Consider the case of $Z^1(a, G)$ and let $R \subseteq Z^1(a, G)^2 \times \tilde{G}$ be defined by

$$R(\alpha, \beta, f) \Leftrightarrow f \cdot \alpha = \beta.$$

It is clearly closed and for each α, β, its (α, β)-section $R_{\alpha,\beta}$ is a translate of $\tilde{G}_\alpha$, thus locally compact and so K_σ. By the Arsenin-Kunugui Theorem (see Kechris [Kec2], 35.46) its projection on $Z^1(a, G)^2$, which is the cohomology relation $\sim$, is Borel. □

We also have:

Proposition 25.3. *Let G be a Polish abelian group. Let $a \in A(\Gamma, X, \mu)$. Then the cohomology relation on $Z^1(a, G)$ is Borel. Similarly for cohomology of equivalence relations.*

Proof. Note that $\tilde{G} \cdot 1 = B^1(a, G)$ is a Borel subgroup of the Polish group $Z^1(a, G)$, and $\sim$ is induced by the cosets of $B^1(a, G)$, so is Borel. □

It now follows from the two preceding results and 23.6 that if G is locally compact admitting an invariant metric or if G is abelian, then the sets $\mathrm{SMOOTH}(a, G)$, $\mathrm{SMOOTH}(E, G)$ are Borel.

One can actually obtain more precise information for $\tilde{G}_1$. First notice that if $f \in \tilde{G}_1$, i.e., $f(\gamma \cdot x) = f(x)$, and $a \in \mathrm{ERG}(\Gamma, X, \mu)$, then f is constant. Thus $a \in \mathrm{ERG}(\Gamma, X, \mu) \Rightarrow \tilde{G}_1 = G \ (\leq \tilde{G})$.

In the case that a is not ergodic, we can determine $\tilde{G}_1$ as follows:

Consider the ergodic decomposition of $a \in A(\Gamma, X, \mu)$, which is given by a Borel a-invariant surjection $\pi : X \to \mathcal{E}$, where $\mathcal{E}$ is the standard Borel space of ergodic invariant measures for a, such that for each $e \in \mathcal{E}, \pi^{-1}(\{e\}) = X_e$, is a-invariant, $e(X_e) = 1$ (and e is the unique invariant measure for $a|X_e$), and if $\nu = \pi_* \mu$, then $\mu = \int e d\nu(e)$. Note now that the map

$$f \mapsto f \circ \pi,$$

$$(L(\mathcal{E}, \nu, G), d_{\mathcal{E},\nu,G}) \to (L(X, \mu, G), d_{X,\mu,G}),$$

is an isometric group embedding, as

$$\begin{aligned}
d_{X,\mu,G}(f \circ \pi, g \circ \pi) &= \int d(f(\pi(x)), g(\pi(x))) d\mu(x) \\
&= \int \left[\int d(f(\pi(x)), g(\pi(x))) de(x) \right] d\nu(e) \\
&= \int d(f(e), g(e)) d\nu(e) \\
&= d_{\mathcal{E},\nu,G}(f, g).
\end{aligned}$$

Thus $L(\mathcal{E}, \nu, G)$ can be identified, via $f \mapsto f \circ \pi$, with a closed subgroup of $L(X, \mu, G)$.

Let now $f \in \tilde{G}_1$. Then again $f(\gamma \cdot x) = f(x)$, so for each $e \in \mathcal{E}, f|X_e$ is constant, say $f(x) = g(e)$, for $x \in X_e$. Then $f = g \circ \pi$, thus $f \in L(\mathcal{E}, \nu, G)$. It follows that

$$\tilde{G}_1 = L(\mathcal{E}, \nu, G).$$

In the special case where G is abelian, the map $f \in L(X, \mu, G) \mapsto f \cdot 1 \in B^1(a, G)$ is a continuous homomorphism of $L(X, \mu, G)$ onto $B^1(a, G)$, so there is a continuous injective homomorphism of the Polish abelian group $L(X, \mu, G)/L(\mathcal{E}, \nu, G)$ onto $B^1(a, G)$ and thus the latter is a Polishable subgroup of $Z^1(a, G)$.

The same results apply to $Z^1(E, G)$ by using the ergodic decomposition of E.

We finally record the following simple facts concerning the stabilizers $\tilde{G}_\alpha$.

Proposition 25.4. *i) Let G be countable, $a \in A(\Gamma, X, \mu)$ and $\alpha \in Z^1(a, G)$. If $f \in \tilde{G}_\alpha$, then $f : X \to C_G(ER(\alpha))$ (where for $H \leq G, C_G(H) =$ the centralizer of H in G). Similarly for equivalence relations.*

ii) Let now G be countable and $a \in \mathrm{WMIX}(\Gamma, X)$ (or just assume that every subgroup of finite index in Γ acts ergodically). Then for any homomorphism $\varphi : \Gamma \to G$ (viewed as an element of $Z^1(a, G)$), $\tilde{G}_\varphi = C_G(\varphi(\Gamma))$.

Proof. i) Let $f \in \tilde{G}_\alpha, g \in ER(\alpha)$. If $h \in G$ is such that $A = f^{-1}(\{h\})$ has positive measure, then for some $x, y \in A, \alpha(x, y) = g$, so $hgh^{-1} = g$, thus $h \in C_G(ER(\alpha))$ and $f : X \to C_G(ER(\alpha))$.

ii) Assume $f \in \tilde{G}_\varphi$. Then as in the proof of 20.1, f is constant, say $f(x) = f_0$ a.e. Then clearly $\forall \gamma (f_0 \varphi(\gamma) f_0^{-1} = \varphi(\gamma))$, i.e., $\tilde{G}_\varphi \leq C_G(\varphi(\Gamma))$ and therefore $\tilde{G}_\varphi = C_G(\varphi(\Gamma))$. □

We conclude with the following problem.

Problem 25.5. *Let $a \in \mathrm{ERG}(\Gamma, X, \mu)$ and let G be countable. Is the cohomology relation $\sim$ on $Z^1(a, G)$ F_σ (as a subset of $Z^1(a, G)^2$)? More generally, for G locally compact admitting an invariant metric, what is the Borel complexity of $\sim$? Similarly for abelian G. Finally, for what $a \in A(\Gamma, X, \mu)$ and Polish G is the cohomology relation $\sim$ on $Z^1(a, G)$ Borel? Similarly for cohomology of equivalence relations.*

26. Cohomology II: The hyperfinite case

We start with the following simple proposition.

Proposition 26.1. *Let E be a finite equivalence relation. Then for any Polish group $G, B^1(E, G) = Z^1(E, G)$.*

Proof. Let T be a Borel transversal for E. Let $\alpha, \beta \in Z^1(E, G)$ in order to show that $\alpha \sim \beta$. Define $f : X \to G$ as follows: For a given $x \in X$, let $\{x_0\} = [x]_E \cap T$. Then put

$$f(x) = \beta(x_0, x)\alpha(x, x_0).$$

(Thus $f(x) = 1$, if $x \in T$.) Then if xEy and $\{x_0\} = T \cap [x]_E = T \cap [y]_E$, we have

$$\begin{aligned}
\beta(x, y) &= \beta(x_0, y)\beta(x, x_0) \\
&= \beta(x_0, y)\beta(x_0, x)^{-1} \\
&= f(y)\alpha(x_0, y)(f(x)\alpha(x_0, x))^{-1} \\
&= f(y)\alpha(x_0, y)\alpha(x, x_0)f(x)^{-1} \\
&= f(y)\alpha(x, y)f(x)^{-1}.
\end{aligned}$$

□

Proposition 26.2. *Let $E_1 \subseteq E_2 \subseteq \dots, E = \bigcup_n E_n, G$ a Polish group. If every cohomology class is dense in each $Z^1(E_n, G)$, the same is true for $Z^1(E, G)$. Similarly, if every $B^1(E_n, G)$ is dense in $Z^1(E_n, G)$, the same is true for $B^1(E, G)$ and $Z^1(E, G)$.*

Proof. Let $\Gamma_1 \leq \Gamma_2 \leq \dots$ be a sequence of countable subgroups of $\mathrm{Aut}(X,\mu)$ such that Γ_n generates E_n. Put $\Gamma_n = \{\gamma_i^{(n)}\}_{i=1}^\infty$. Let $\mu_i^{(n)}$ be the copy of μ on graph$(\gamma_i^{(n)})$ via the map $x \mapsto (x, \gamma_i^{(n)}(x))$. Let M be the usual measure on $E, M(A) = \int \mathrm{card}(A_x) d\mu(x)$, and M_n the similar measure for E_n, so that $M_n = M|E_n$. Note that $M|E_n \sim \sum_{i=1}^\infty 2^{-i}\mu_i^{(n)}, M \sim \sum_{i=1}^\infty \sum_{m=1}^\infty 2^{-(i+m)}\mu_i^{(m)}$.

Let now $\alpha, \beta \in Z^1(E,G)$. Consider $\alpha|E_n, \beta|E_n$. Since

$$\sum_{m\leq n}\sum_{i=1}^\infty 2^{-(i+m)}\mu_i^{(m)} \sim M|E_n,$$

there is $f_n : X \to G$ such that

$$\sum_{m\leq n}\sum_{i=1}^\infty 2^{-(i+m)} \int d(f_n\cdot\alpha,\beta)d\mu_i^{(m)} \leq \frac{1}{n},$$

where $d \leq 1$ is a compatible metric for G. Therefore

$$\sum_{i,m} 2^{-(i+m)} \int d(f_n\cdot\alpha,\beta)d\mu_i^{(m)} \leq \sum_{m\leq n} 2^{-m}\left(\sum_{i=1}^\infty 2^{-i}\int d(f_n\cdot\alpha,\beta)d\mu_i^{(m)}\right)$$
$$+\sum_{m>n}2^{-m} \leq \frac{1}{n} + \sum_{m>n} 2^{-m} \to 0$$

as $n \to \infty$. So $f_n \cdot \alpha \to \beta$ in $Z^1(E,G)$.

For the case of $B^1(E,G), Z^1(E,G)$ apply the preceding argument to $\alpha = 1$. □

Corollary 26.3 (Parthasarathy-Schmidt [PS]). *If E is hyperfinite and G is Polish, then every cohomology class is dense in $Z^1(E,G)$.*

Moreover we have the following result, certain cases of which were proved in Hjorth [Hj1].

Theorem 26.4. *Let E be ergodic, hyperfinite, and let $G \neq \{1\}$ be a Polish group. Then $B^1(E,G) \neq Z^1(E,G)$. If moreover G admits an invariant metric, then every cohomology class is dense and meager in $Z^1(E,G)$ and so* $\mathrm{ROUGH}(E,G) = Z^1(E,G) = \overline{B^1(E,G)}$.

Proof. For the first assertion, assume first that G is not compact. Then by Solecki [So] there is a free continuous action of G on a Polish space Y and a Borel reduction $p : X \to Y$ of E to E_G^Y. Let $\alpha : E \to G$ be the corresponding cocycle, $\alpha(x,y)\cdot p(x) = p(y)$. Then α is not in $B^1(E,G)$, since if $\alpha(x,y) = f(y)f(x)^{-1}$, then $q(x) = f(x)^{-1}\cdot p(x)$ is E-invariant, so $q(x) = y_0 \in Y$, a.e., i.e., p maps into a single orbit of G, a.e., a contradiction.

When $G \neq \{1\}$ is compact, one can use for example the fact that there is a cocycle $\alpha : E \to G$ with $ER(\alpha) = G$ (see Zimmer [Zi1]). Then it is easy to see that $\alpha \notin B^1(E,G)$.

For the second assertion, we have, by Section 23, that $B^1(E,G)$ is in ROUGH(E,G) and so all cohomology classes in $\overline{B^1(E,G)} = Z^1(E,G)$ are dense and meager. □

In the next section, we will see more general facts that also imply that the action of $L(X,\mu,G)$ on $Z^1(E,G)$ is turbulent, when G admits an invariant metric (see 27.4).

We will next prove a converse result.

Theorem 26.5 (Kechris). *Let E be given by a free action of a countable group. Then the following are equivalent:*

(i) E is hyperfinite,

(ii) For every countable $G, \overline{B^1(E,G)} = Z^1(E,G)$,

(iii) For every Polish $G, \overline{B^1(E,G)} = Z^1(E,G)$.

Proof. We have seen that (i) ⇒ (iii) ⇒ (ii). We will now prove that (ii) ⇒ (i).

Lemma 26.6. *Let $a \in A(\Gamma, X, \mu)$ and let $\varphi : \Gamma \to G$ be a homomorphism, where G is a countable group. Then if $\varphi \in \overline{B^1(a,G)}, \varphi(\Gamma)$ is amenable.*

Proof. Since $\varphi \in \overline{B^1(a,G)}$, there is a sequence $f_n \in L(X,\mu,G)$ such that

$$\forall\gamma\forall^\infty n(f_n(\gamma\cdot x) = \varphi(\gamma)f_n(x)), \text{ a.e.}(x).$$

Then define $\Phi : \ell^\infty(G) \to \mathbb{C}$ by

$$\Phi(F) = \int \Phi_{\mathbb{N}}(\{F(f_n(x))\}_n)d\mu(x),$$

where $\Phi_{\mathbb{N}} : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ is a shift-invariant mean (i.e., satisfies $\Phi_{\mathbb{N}}(\{x_n\}) = \Phi_{\mathbb{N}}(\{x_{n+1}\})$ which is universally measurable, so that the above integral makes sense. We are using here the theorem of Mokobodzki that asserts the existence of such means under the *Continuum Hypothesis* (CH) (see, e.g., Kechris [Kec1]). Since the statement of the result we want to prove is sufficiently absolute, it is well-known that we can eliminate the use of CH by the usual metamathematical methods. See, e.g., Adams-Lyons [AL] for an exposition of this metamathematical technique.

It is not hard to see now that Φ is $\varphi(\Gamma)$-invariant under the translation action of $\varphi(\Gamma)$ on G, thus $\varphi(\Gamma)$ is amenable. □

So let $E = E_\Gamma^X$, where Γ is a countable group acting freely on X. Call this action a. Let $\varphi : \Gamma \to \Gamma$ be the identity. Then $Z^1(a,\Gamma)$ can be identified with $Z^1(E,\Gamma)$ and similarly for $B^1(a,\Gamma)$ and $B^1(E,\Gamma)$, and we have that $\varphi \in \overline{B^1(a,\Gamma)}$, so $\varphi(\Gamma) = \Gamma$ is amenable and thus E is hyperfinite. □

I do not know if the hypothesis that E is given by a free action is needed in 26.5.

Before we proceed to the next result, it would be convenient to state a characterization of the cocycles in $\overline{B^1(E,G)}$ for an arbitrary equivalence relation E and countable group G.

For each countable group G, consider the equivalence relation on $G^{\mathbb{N}}$ given by

$$\{g_n\}E_0(G)\{h_n\} \Leftrightarrow \forall^\infty n(g_n = h_n).$$

Clearly $E_0(G)$ is Borel hyperfinite. Let

$$X_0(G) = G^{\mathbb{N}}/E_0(G).$$

The group G acts freely on $X_0(G)$ by diagonal (left-) translation

$$g \cdot [\{g_n\}] = [\{gg_n\}].$$

The equivalence relation $E_G^{X_0(G)}$, when lifted up to $G^{\mathbb{N}}$, becomes the equivalence relation

$$\{g_n\}F_{t,0}(G)\{h_n\} \Leftrightarrow \exists g\forall^\infty n(gg_n = h_n).$$

Note that $F_{t,0}(G)$ is hyperfinite. Indeed $F_{t,0}(G) = \bigcup_{m=1}^\infty F_m$, where

$$\{g_n\}F_m\{h_n\} \Leftrightarrow \exists g\forall n \geq m(gg_n = h_n).$$

Let G act (diagonally) by translation on $G^{\mathbb{N}}$

$$g \cdot \{g_n\} = \{gg_n\}.$$

Denote by $F_t(G)$ the corresponding equivalence relation. It is clearly smooth as $\{1\} \times G^{\mathbb{N}}$ is a Borel transversal. Now F_m is Borel reducible to $F_t(G)$, so it is smooth as well. Since $F_1 \subseteq F_2 \subseteq \dots, F_{t,0}(G)$ is hyperfinite (see Dougherty-Jackson-Kechris [DJK], 5.1).

Given now an equivalence relation E on (X, μ) and a homomorphism $p : X \to X_0(G)$ of E into $E_G^{X_0(G)}$, a.e., with Borel lifting, we have an associated cocycle $\alpha \in Z^1(E, G)$ given by $\alpha(x, y) \cdot p(x) = p(y)$. Equivalently, by lifting, for any Borel homomorphism q of E into $F_{t,0}(G)$, a.e., we have an associated cocycle $\alpha \in Z^1(E, G)$ defined by $\alpha(x, y) =$ the unique $g \in G$ such that $\forall^\infty n(gq(x)_n = q(y)_n)$, where $q(x) = \{q(x)_n\}$. We now have:

Proposition 26.7. *Let E be an equivalence relation on (X, μ) and G a countable group. Then $\alpha \in \overline{B^1(E, G)}$ iff α is associated to an a.e. Borel homomorphism of E into $F_{t,0}(G)$.*

Proof. $\Rightarrow$: If $\alpha \in \overline{B^1(E, G)}$, we can find Borel $q_n : X \to G$ such that, a.e.,

$$xEy \Rightarrow \forall^\infty n(q_n(y) = \alpha(x, y)q_n(x)).$$

Let $q(x) = \{q_n(x)\}, q : X \to G^{\mathbb{N}}$. Then clearly q is a Borel homomorphism of E into $F_{t,0}(G)$ with associated cocycle α.

$\Leftarrow$: Let $q : X \to G^{\mathbb{N}}$ be a Borel homomorphism of E into $F_{t,0}(G)$ with associated cocycle α and let $q = \{q_n\}$. Then, a.e., $xEy \Rightarrow \forall^\infty n(q_n(y) = \alpha(x, y)q_n(x))$, so $\alpha \in \overline{B^1(E, G)}$. □

In the case of actions $a \in A(\Gamma, X, \mu)$, we say that $\alpha \in Z^1(a, G)$ is associated to a Borel homomorphism of a into $F_{t,0}(G)$ if there is a Borel map $q : X \to G^{\mathbb{N}}$ such that, a.e., $q(\gamma \cdot x)F_{t,0}(G)q(x)$ and $\alpha(\gamma, x) =$ the unique $g \in G$ such that $\forall^\infty n(gq(x)_n = q(\gamma \cdot x)_n)$. Then the analog of 26.7 goes through.

We now have the following consequence.

Proposition 26.8 (Kechris). *Let E be E_0-ergodic and G be a countable group. Then $B^1(E,G)$ is closed. Similarly for actions.*

Proof. Let $\alpha \in \overline{B^1(E,G)}$. Then α is associated to a homomorphism q of E into $F_{t,0}(G)$. Since E is E_0-ergodic and $F_{t,0}(G)$ is hyperfinite, $q = \{q_n\}$ maps into a single $F_{t,0}(G)-$class, a.e., say that of $\{h_n^0\}$. Thus

$$\exists g \forall^\infty n(q_n(x) = gh_n^0), \text{ a.e.}(x),$$

so there is $g : X \to G$ such that

$$\forall^\infty n(q_n(x) = g(x)h_n^0), \text{ a.e.}(x).$$

Thus we have, on a set of measure 1,

$$\begin{aligned} xEy \Rightarrow \forall^\infty n(q_n(y) &= \alpha(x,y)q_n(x) \\ &= g(y)h_n^0 \\ &= \alpha(x,y)g(x)h_n^0), \end{aligned}$$

therefore

$$xEy \Rightarrow \alpha(x,y) = g(y)g(x)^{-1},$$

i.e., $\alpha \in B^1(E,G)$ and $B^1(E,G)$ is closed. □

Schmidt has proved 26.8 when G is an abelian locally compact Polish group; see Schmidt [Sc5], 3.3.

We now have the following result that generalizes a theorem of Schmidt (see Schmidt [Sc3], Remark (3.5)), who dealt with the case of abelian G.

Theorem 26.9 (Kechris). *Let Γ be a countable group. Then the following are equivalent:*

(i) Γ *is amenable,*

(ii) $\forall a \in \mathrm{FR}(\Gamma, X, \mu) \forall$*Polish* G,

$$\overline{B^1(a,G)} = Z^1(a,G),$$

(iii) $\forall a \in \mathrm{FR}(\Gamma, X, \mu) \forall$*countable* G,

$$\overline{B^1(a,G)} = Z^1(a,G),$$

(iv) $\forall a \in \mathrm{FRERG}(\Gamma, X, \mu) \forall$*Polish* $G \neq \{1\}$,

$$B^1(a,G) \neq \overline{B^1(a,G)} = Z^1(a,G),$$

(v) $\forall a \in \mathrm{FRERG}(\Gamma, X, \mu) \exists$*countable* G,

$$B^1(a,G) \neq \overline{B^1(a,G)}.$$

Proof. (i) ⇒ (ii) by 26.3. (i) ⇒ (iv) by 26.4. (ii) ⇒ (iii) and (iv) ⇒ (v) are obvious. Finally, we prove ¬(i) ⇒ ¬(iii), ¬(v).

Assume that Γ is not amenable. Then the shift action s of Γ on 2^Γ is free and E_0-ergodic (see, e.g. Hjorth-Kechris [HK3], A4.1). Then $B^1(s,G)$ is closed by 26.8, for any countable G, and $B^1(s,\Gamma) \neq Z^1(s,\Gamma)$ by 26.6. □

Comments. There is an extensive literature on the structure of cocycles of hyperfinite ergodic equivalence relations E, especially in the works of Bezuglyi, Danilenko, Fedorov, Golodets, Schmidt, Sinelshchikov and Zimmer. For example, it is known that the Polish locally compact groups G that are essential ranges of such cocycles are exactly the amenable ones. Given such a G, the set of cocycles $\alpha \in Z^1(E,G)$ such that $ER(\alpha) = G$ is comeager in $Z^1(E,G)$. Every ergodic, quasi-invariant action of an amenable locally compact Polish group is isomorphic to a Mackey action (in the sense of Section 21, **(C)**). Every cocycle $\alpha \in Z^1(E,G), G$ Polish, is cohomologous to a cocycle β taking values in any given countable dense subgroup of G. Finally, extending Dye's Theorem, for any Polish group G, if $\alpha,\beta \in Z^1(E,G)$ have essential range G, then α,β are weakly equivalent, $\alpha \sim^w \beta$. Since we are primarily interested here in non-hyperfinite equivalence relations, we will not pursue the hyperfinite case any further. For a recent survey of results in this case, see Bezuglyi [Be].

27. Cohomology III: The non E_0-ergodic case

(A) We start with the following result:

Proposition 27.1 (Kechris). *Let E be ergodic but not E_0-ergodic and let G be a Polish group admitting an invariant metric. If $\alpha \in Z^1(E,G)$ has non-trivial stabilizer for the action of $\tilde{G}$ on $Z^1(E,G)$, i.e., $\tilde{G}_\alpha \neq \{1\}$, then $\alpha \in \mathrm{ROUGH}(E,G)$. Similarly for cocycles of actions.*

Proof. Fix $g_0 \in \tilde{G}_\alpha, g_0 \neq 1$. Recall also from the proof of 25.1 that for $f,g \in \tilde{G}_\alpha$, we have $d(f(x),g(x)) = \tilde{d}(f,g)$, a.e., where $d \leq 1$ is an invariant metric on G.

Fix also a countable group Γ and a Borel action of Γ on X with $E = E^X_\Gamma$. Since E is not E_0-ergodic, there is a sequence $\{A_n\}$ of Borel sets with $\mu(\gamma \cdot A_n \triangle A_n) \to 0, \forall \gamma \in \Gamma$, and $\mu(A_n) = 1/2$. Let $\chi_n \in \tilde{G}$ be defined by $\chi_n(x) = g_0(x)$, if $x \in A_n, \chi_n(x) = 1$, if $x \notin A_n$.

Claim. $\chi_n \cdot \alpha \to \alpha$.

Granting this claim, if $\alpha \in \mathrm{SMOOTH}(E,G)$, towards a contradiction, then the orbit $\tilde{G}\cdot\alpha = [\alpha]_\sim$ is closed, so by Effros' Theorem, writing $H = \tilde{G}_\alpha$, we have that $\chi_n H \to H$ in $\tilde{G}/H$, thus we can find $h_n \in H$ with $\tilde{d}(\chi_n,h_n) = \int d(\chi_n(x),h_n(x))d\mu(x) \to 0$. Clearly $d(\chi_n(x),h_n(x)) = d(g_0(x),h_n(x))$, if $x \in A_n$; $= d(1,h_n(x))$, if $x \notin A_n$. But $1,g_0,h_n \in H$, so

$$\tilde{d}(g_0,h_n) = d(g_0(x),h_n(x)),$$
$$\tilde{d}(1,h_n) = d(1,h_n(x)),$$

a.e., thus

$$d(\chi_n(x), h_n(x)) = \begin{cases} \tilde{d}(g_0, h_n), & \text{if } x \in A_n, \\ \tilde{d}(1, h_n), & \text{if } x \notin A_n, \end{cases}$$

a.e., and so

$$\tilde{d}(\chi_n, h_n) = \frac{1}{2}(\tilde{d}(g_0, h_n) + \tilde{d}(1, h_n)) \to 0,$$

i.e., $h_n \to g_0, h_n \to 1$, a contradiction.

Proof of the Claim. We need to show that for each γ,

$$(*) \qquad \int d(\chi_n(\gamma^{-1} \cdot x)\alpha(x, \gamma^{-1} \cdot x)\chi_n(x)^{-1}, \alpha(x, \gamma^{-1} \cdot x))d\mu(x)$$

converges to 0 as $n \to \infty$. But

$$\chi_n(\gamma^{-1} \cdot x) = \begin{cases} g_0(\gamma^{-1} \cdot x), & \text{if } x \in \gamma \cdot A_n, \\ 1, & \text{if } x \notin \gamma \cdot A_n, \end{cases}$$

so $d(\chi_n(\gamma^{-1} \cdot x)\alpha(x, \gamma^{-1} \cdot x)\chi_n(x)^{-1}, \alpha(x, \gamma^{-1} \cdot x)) \leq 1$, if $x \in \gamma \cdot A_n \triangle A_n$, and otherwise it is equal to $d(g_0(\gamma^{-1} \cdot x)\alpha(x, \gamma^{-1} \cdot x)g_0(x)^{-1}, \alpha(x, \gamma^{-1} \cdot x)) = 0$, if $x \in \gamma \cdot A_n \cap A_n$ (as $g_0 \cdot \alpha = \alpha$) and to $d(\alpha(x, \gamma^{-1} \cdot x), \alpha(x, \gamma^{-1} \cdot x)) = 0$, if $x \notin \gamma \cdot A_n \cup A_n$. Thus $(*) \leq \mu(\gamma \cdot A_n \triangle A_n) \to 0$. □

Corollary 27.2. *For any E which is ergodic but not E_0-ergodic and any Polish group $G \neq \{1\}$ admitting an invariant metric,*

$$B^1(E, G) \subseteq \text{ROUGH}(E, G).$$

Similarly for actions.

Proof. $\tilde{G}_1 = G \neq \{1\}$. □

Thus for any ergodic, non E_0-ergodic E and non-trivial G admitting an invariant metric (in particular countable G), the rough part ROUGH(E, G) of $Z^1(E, G)$ is non-empty. Of course SMOOTH(E, G) can be empty and ROUGH(G) can contain a single cohomology class closure, as for example when E is hyperfinite. We will discuss later in this section various examples where ROUGH(E, G) contains many cohomology class closures and similarly for SMOOTH(E, G).

We have seen in Section 23, that the action of $\tilde{G}$ on each cohomology class closure C contained in ROUGH(E, G) is minimal and every cohomology class in C is meager in C. We actually have the following result.

Theorem 27.3 (Kechris). *Suppose E is not E_0-ergodic and let G be a Polish group admitting an invariant metric. Let $\alpha \in$ ROUGH(E, G). Then the action of $\tilde{G}$ on the closure of $[\alpha]_\sim$ is turbulent. Similarly for actions.*

Proof. The argument is reminiscent of that in the proofs of 5.2 and 13.3. We only need to verify that α is a turbulent point for this action. Fix a countable group $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ and a Borel action of Γ on X with $E_\Gamma^X = E$. Consider the open nbhd

$$U_{\alpha,\epsilon} = \{\beta : \tilde{d}^1(\alpha,\beta) = \sum_k 2^{-k} \int d(\alpha(x,\gamma_k\cdot x),\beta(x,\gamma_k\cdot x))d\mu(x) < \epsilon\},$$

where $\epsilon > 0$ and $d \leq 1$ is a compatible invariant metric for G. Let $f \in \tilde{G}$ be such that $\beta = f\cdot\alpha \in U_{\alpha,\epsilon}$. It is enough to find a continuous path $\lambda \mapsto f_\lambda, \lambda \in [0,1]$, from 1 to f in $\tilde{G}$ with $f_\lambda \cdot \alpha$ lying in $U_{\alpha,\epsilon}, \forall\lambda$.

Let $E_0 = \bigcup_{n\geq 1} E_n$ on $Y = 2^{\mathbb{N}}$, where each E_n is a finite Borel equivalence relation and $E_1 \subseteq E_2 \subseteq \dots$ Since E is not E_0-ergodic, let $\pi : X \to Y$ be a Borel homomorphism of E into E_0 such that $\mu(\pi^{-1}(C)) = 0$ for every E_0-class C.

Say $\tilde{d}^1(\alpha,\beta) = \rho < \epsilon$ and choose N_0 large enough so that $\sum_{k>N_0} 2^{-k} < \frac{(\epsilon-\rho)}{3}$. Now for almost all x,

$$\exists n \geq 1 \forall k \leq N_0 (\pi(x) E_n \pi(\gamma_k \cdot x) E_n \pi(\gamma_k^{-1}\cdot x)).$$

So if

$$X_n = \{x : \forall k \leq N_0(\pi(x)E_n\pi(\gamma_k\cdot x))\},$$

then $X_1 \subseteq X_2 \subseteq \dots$ and $\mu(X_n) \to 1$. So choose $N \geq N_0$ large enough so that

$$\mu(\sim X_N) < \frac{(\epsilon-\rho)}{3}.$$

Then as in the proof of 5.2 we can find a continuous map $\lambda \mapsto X_\lambda$ from [0,1] to MALG_μ with $X_0 = \emptyset, X_1 = X, \lambda \leq \lambda' \Rightarrow X_\lambda \subseteq X_{\lambda'}$ & $\mu(X_{\lambda'} \setminus X_\lambda) \leq \lambda' - \lambda$, and X_λ $\pi^{-1}(E_N)$-invariant, so that, in particular, if $k \leq N_0$ and $x \in X_N$, then $x \in X_\lambda \Leftrightarrow \gamma_k \cdot x \in X_\lambda$.

Let $f_\lambda(x) = f(x)$, if $x \in X_\lambda$; $= 1$, if $x \notin X_\lambda$. We claim that $f_\lambda\cdot\alpha \in U_{\alpha,\epsilon}$. Since $f_0 = 1, f_1 = f$ and $\lambda \mapsto f_\lambda$ is continuous, this will complete the proof.

Now $\tilde{d}^1(\alpha, f_\lambda\cdot\alpha)$ is clearly bounded by

$$\sum_{k\leq N_0} 2^{-k} \int d(\alpha(x,\gamma_k\cdot x), f_\lambda(\gamma_k\cdot x)\alpha(x,\gamma_k\cdot x)f_\lambda(x)^{-1})d\mu(x)$$

with an error at most $\frac{(\epsilon-\rho)}{3}$. We can break this sum into four parts, restricting the integral to the sets $X_\lambda \cap X_N, X_\lambda \setminus X_N, X_N \setminus X_\lambda, \sim X_\lambda \cap \sim X_N$, resp. The first part is bounded by $\tilde{d}^1(\alpha,\beta) = \rho$, since $f_\lambda = f$ on X_λ. The second is bounded by $\mu(\sim X_N) < \frac{(\epsilon-\rho)}{3}$, since $d \leq 1$. The third is 0 since $f_\lambda = 1$ on $\sim X_\lambda$. Finally the last one is again bounded by $\mu(\sim X_N)$. Thus $\tilde{d}^1(\alpha, f_\lambda\cdot\alpha)$ is less than $\frac{(\epsilon-\rho)}{3} + \rho + \frac{(\epsilon-\rho)}{3} + 0 + \frac{(\epsilon-\rho)}{3} = \epsilon$. □

Remark. This argument actually shows that for *any* Polish group G and any E which is not E_0-ergodic, every $\alpha \in Z^1(E,G)$ is a turbulent point for the action of $\tilde{G}$ on the closure of the cohomology class of α.

Corollary 27.4. *Suppose E is ergodic but not E_0-ergodic and let $G \neq \{1\}$ be a Polish group admitting an invariant metric. Then the action of $\tilde{G}$ on*

$\overline{B^1(E,G)}$ *is turbulent. In particular, the cohomology relation on* $\overline{B^1(E,G)}$ *(and thus also on* $Z^1(E,G)$*) does not admit classification by countable structures. Similarly for actions.*

We should point out here that Hjorth [Hj1] has proved certain cases of 27.4 by different methods for hyperfinite E.

(B) We will now discuss some implications of the results in **(A)** to cocycle superrigidity.

Let $a \in A(\Gamma, X, \mu)$ and let $\mathcal{G}$ be a class of groups. Then a is called *$\mathcal{G}$-cocycle superrigid* if every $\alpha \in Z^1(a,G)$, where $G \in \mathcal{G}$, is cohomologous to a homomorphism $\varphi : \Gamma \to G$. This concept arose in recent work of Popa, where many important examples of actions that are $\mathcal{G}_{count}$-cocycle superrigid were discovered, where $\mathcal{G}_{count}$ = the class of all countable groups. For instance, Popa [Po2] shows (among other things) that the shift action of Γ on $[0,1]^\Gamma$, for various kinds of groups Γ, is $\mathcal{G}_{count}$-superrigid. When $\mathcal{G} = \{G\}$ we say that a is *G-cocycle superrigid.*

We now have the following result which shows that superrigidity entails E_0-ergodicity.

Corollary 27.5. *Let* $a \in \mathrm{WMIX}(\Gamma, X, \mu)$ *(in fact it is enough to assume that every finite index subgroup* Γ *acts ergodically) and* $a \notin \mathrm{E_0RG}(\Gamma, X, \mu)$*. Then for any countable group* $G \neq \{1\}$*, a is not G-cocycle superrigid. In fact, for every* $\alpha \in \mathrm{ROUGH}(a,G)$*, the generic cocycle in the closure of the cohomology class of α is not cohomologous to a homomorphism.*

Proof. Let $\alpha \in \mathrm{ROUGH}(a,G), C = \overline{[\alpha]_\sim}$. Then the action of $\tilde{G}$ on C is turbulent. Let $A = \{\beta \in C : \beta$ is cohomologous to a homomorphism$\}$. Since the action of $\tilde{G}$ on C is minimal, A is either meager or comeager in C. Assume towards a contradiction that A is comeager. Then by going if necessary to comeager subset of A, we can assume that there is a Borel function $\Phi : A \to \mathrm{Hom}(\Gamma, G)$ (viewed as a Polish space with the product topology) such that $\alpha \sim \Phi(\alpha)$. But then, by 20.1, $\alpha \sim \beta \Leftrightarrow \Phi(\alpha) \sim \Phi(\beta) \Leftrightarrow \Phi(\alpha), \Phi(\beta)$ are conjugate. Clearly conjugacy in $\mathrm{Hom}(\Gamma, G)$ is induced by a countable group action, so $\sim$ on A is classifiable by countable structures, contradicting turbulence. □

We also have the following consequence concerning generic actions.

Corollary 27.6. *Suppose the group* Γ *does not have property* (T)*. Then the generic action of* Γ *is not G-cocycle superrigid, for any countable group* $G \neq \{1\}$.

Proof. The generic action of Γ is weak mixing and non E_0-ergodic (by 12.3 and 12.9). □

We will next see some characterizations of the class of property (T) groups, related to superrigidity.

Hjorth-Kechris [HK2], B.4 and (independently) Jolissaint [Jo2] proved that if Γ has property (T), and G is a Polish locally compact group with

HAP, then for every $a \in \mathrm{ERG}(\Gamma, X, \mu)$, every cocycle $\alpha \in Z^1(a, G)$ is cohomologous to a cocycle β which takes values in a compact subgroup of G. (See also Connes-Jones [CJ2] for earlier related results in the context of operator algebras.) In particular, if G contains no non-trivial such subgroups (e.g., G countable, torsion free), then $\alpha \in B^1(a, G)$, so a is also G-cocycle superrigid. We note the following converse.

Corollary 27.7. *Let Γ be a countable group. Then the following are equivalent:*

(i) Γ does not have property (T).

(ii) For every countable group $G \neq \{1\}$, the generic ergodic action of Γ is not G-cocycle superrigid.

(iii) There is a countable, torsion-free group G with HAP *and an ergodic action of Γ which is not G-cocycle superrigid.*

Proof. (i) $\Rightarrow$ (ii) by 27.6. (ii) $\Rightarrow$ (iii) is obvious. Finally $\neg$(i) $\Rightarrow$ $\neg$(iii) by the preceding paragraph. □

Call now a countable group Γ *somewhere cocycle superrigid* if every ergodic action of Γ is G-superrigid for some countable $G \neq \{1\}$, perhaps depending on the action. Denote by $\mathcal{G}^0_{\mathrm{HAP}}$ the class of Polish locally compact groups that have HAP and contain no non-trivial compact subgroups. Finally we say that Γ is *$\mathcal{G}$-cocycle superrigid* if every ergodic action of Γ is *$\mathcal{G}$-cocycle superrigid.*

Corollary 27.8. *Let $\mathcal{F}$ be a class of countable groups. Then the following are equivalent:*

(i) $\mathcal{F}$ is contained in the class of property (T) *groups,*

(ii) $\forall \Gamma \in \mathcal{F}(\Gamma$ is somewhere cocycle superrigid$)$,

(iii) $\forall \Gamma \in \mathcal{F}(\Gamma$ is $\mathcal{G}^0_{\mathrm{HAP}}$-cocycle superrigid$)$.

Proof. (i) $\Rightarrow$ (iii) has been already discussed and (iii) $\Rightarrow$ (ii) is obvious. Assume now (i) fails and let $\Gamma \in \mathcal{F}$ fail to have property (T). Then by 27.6, the generic action a of Γ is not G-cocycle superrigid, for any countable $G \neq \{1\}$, so (ii) fails. □

Thus the class of property (T) groups is the largest class of countable groups *all* of whose ergodic actions have some non-trivial cocycle superrigidity.

(C) As in Section 15, the turbulence results 27.3, 27.4 above have also consequences concerning path connectedness in the space of cocycles.

Theorem 27.9 (Kechris). *Suppose E is not E_0-ergodic and let G be a Polish group admitting an invariant metric. Then for any $\alpha \in Z^1(E, G)$ the closure of the cohomology class of α is path connected. Similarly for actions.*

Proof. If $\alpha \in \mathrm{SMOOTH}(E, G)$, this is clear as $\overline{[\alpha]_\sim} = \tilde{G} \cdot \alpha$ and $\tilde{G}$ is path connected. If $\alpha \in \mathrm{ROUGH}(E, G)$, then this follows from the proof of 27.3 and the proof of 15.2. □

We also have the following.

Theorem 27.10 (Kechris). *For any equivalence relation E on (X, μ) and countable G, $\overline{B^1(E,G)}$ is path connected. Similarly for actions.*

Proof. If E is not E_0-ergodic, this follows from 27.9. If E is E_0-ergodic, then by 26.8, $\overline{B^1(E,G)} = B^1(E,G) = \tilde{G} \cdot 1$, and we are done. □

Corollary 27.11. *For any Γ and countable group G,*

$$\{(a, \alpha) \in AZ^1(\Gamma, X, \mu, G) : \alpha \in \overline{B^1(a,G)}\}$$

is path connected.

Proof. Fix (a, α) in this set. Then α is path connected to 1 with a path in $\overline{B^1(a,G)}$. So it is enough to show that $A(\Gamma, X, \mu) \times \{1\}$ is path connected, which follows from 15.11. □

Corollary 27.12. *If Γ is amenable, then $AZ^1(\Gamma, X, \mu, G)$ is path connected, for any countable group G.*

(D) We have already seen that when E is ergodic but not E_0-ergodic and $G \neq \{1\}$ is countable, $\text{ROUGH}(E, G)$ is always non-empty, while if E is moreover hyperfinite, then $\text{ROUGH}(E, G) = Z^1(E, G)$. We also have:

Proposition 27.13. *Let Γ be a countable group. Then for any action $a \in \text{ERG}(\Gamma, X, \mu) \setminus \text{E}_0\text{RG}(\Gamma, X, \mu)$ and any weakly commutative, nontrivial, countable group G, $\text{ROUGH}(a, G) = Z^1(a, G)$.*

Proof. Let $\alpha \in Z^1(a, G)$, where $a \in \text{ERG}(\Gamma, X, \mu) \setminus \text{E}_0\text{RG}(\Gamma, X, \mu)$. We want to show that $\alpha \in \text{ROUGH}(a, G)$. By 27.1, we can assume that the stabilizer $\tilde{G}_\alpha$ is trivial. Thus by Effros' Theorem it is enough to find $g_n \in \tilde{G}$ such that $g_n \cdot \alpha \to \alpha$ but $g_n \not\to 1$. Let $\{g_n\} \subseteq G$ witness the weak commutativity of G, i.e., $g_n \neq 1$ and $\forall g \in G \forall^\infty n(g g_n = g_n g)$. Then clearly $g_n \cdot \alpha \to \alpha$. On the other hand $g_n \not\to 1$ in $\tilde{G}$. □

This should be contrasted with the following result.

Proposition 27.14 (Kechris). *Assume that Γ is a group that is not inner amenable and let $a \in \text{ERG}(\Gamma, X, \mu)$. Let $\alpha_0(\gamma, x) = \gamma$, so that $\alpha_0 \in Z^1(a, \Gamma)$. Then $\alpha_0 \in \text{SMOOTH}(a, \Gamma)$, thus $\text{ROUGH}(a, \Gamma) \neq Z^1(a, \Gamma)$.*

Proof. First note that $\tilde{\Gamma}_{\alpha_0} = \{1\}$. Indeed, if $f \cdot \alpha_0 = \alpha_0$, then $f(\gamma \cdot x) = \gamma f(x) \gamma^{-1}$, so $f : X \to \Gamma$ is a homomorphism of the Γ-action on X to the conjugacy action on Γ. Thus $f_* \mu$ is an ergodic, invariant measure for this last action, so, since Γ is not inner amenable, it concentrates on $\{1\}$, i.e., $f = 1$.

Thus, by Effros' Theorem and 23.5, it is enough to show that if $f_n \in \tilde{\Gamma}$ and $f_n \cdot \alpha_0 \to \alpha_0$, then $f_n \to 1$. Equivalently, it is enough to show that if $\forall \gamma \forall^\infty n(f_n(\gamma \cdot x)\gamma f_n(x)^{-1} = \gamma)$, a.e.$(x)$, then $f_n \to 1$. So assume that

$$(*) \qquad \forall \gamma \forall^\infty n(f_n(\gamma \cdot x) = \gamma f_n(x) \gamma^{-1}), \text{ a.e.}(x),$$

but $f_n \not\to 1$, towards a contradiction. Then the set $A = \{x : \exists^\infty n(f_n(x) \neq 1)\}$ has positive measure, but it is also Γ-invariant by (*), thus, by ergodicity,

$$(**) \qquad \exists^\infty n(f_n(x) \neq 1), \text{ a.e.}(x).$$

Put

$$\varphi(x) = \{f_n(x) : n \in \mathbb{N}\}.$$

Claim. *$\varphi(x)$ is finite, a.e.(x).*

We assume this claim and complete the proof. Let

$$\Psi(x) = \{g \in \varphi(x) : g \neq 1 \ \& \ \exists^\infty n(f_n(x) = g)\}.$$

Thus by (**), $1 \notin \Psi(x) \neq \emptyset$. Next we check that

$$\Psi(\gamma \cdot x) = \gamma \Psi(x) \gamma^{-1}.$$

Indeed fix γ, x and by (*) let N be such that $\forall n \geq N(f_n(\gamma \cdot x) = \gamma f_n(x)\gamma^{-1})$. If $g \in \Psi(x)$, let $N < n_0 < n_1 < \dots$ be such that $f_{n_i}(x) = g$. Then $\gamma g \gamma^{-1} = \gamma f_{n_i}(x)\gamma^{-1} = f_{n_i}(\gamma \cdot x)$, so $\gamma g \gamma^{-1} \in \Psi(\gamma \cdot x)$.

Thus in particular $x \mapsto \text{card}(\Psi(x))$ is Γ-invariant, so constant a.e., say equal to $K \geq 1$. Then $\Psi(x)$ is a homomorphism of the Γ-action on X into the conjugacy action of Γ on the set $\mathcal{P}_K(\Gamma)$ of subsets of Γ of cardinality K. Thus $\Psi_*\mu$ is an invariant, ergodic measure for this action, so concentrates on a single finite orbit, thus

$$\{\gamma \Psi(x) \gamma^{-1} : \gamma \in \Gamma\} \text{ is finite, a.e.}(x).$$

Now fix x and $g_0 \in \Psi(x)$. Then $C_0 = \{\gamma g_0 \gamma^{-1} : \gamma \in \Gamma\}$ is infinite (as Γ is ICC), but $C_0 \subseteq \bigcup_{\gamma \in \Gamma} \gamma \Psi(x) \gamma^{-1}$ which is finite, a contradiction.

Proof of the Claim. Let $E(x)$ be the equivalence relation on $\mathbb{N}$ given by

$$mE(x)n \Leftrightarrow f_m(x) = f_n(x).$$

Then, by (*), $\forall \gamma \exists N \forall m, n \geq N[mE(x)n \Leftrightarrow mE(\gamma \cdot x)n]$, a.e.$(x)$. In particular, $\varphi(x)$ is finite iff $\varphi(\gamma \cdot x)$ is finite, so, towards a contradiction, we can assume that $\varphi(x)$ is infinite, a.e.(x). Let then $n_1(x) < n_2(x) < \dots$ enumerate in increasing order the least elements of the $E(x)$-classes at which $f_n(x) \neq 1$. Again we must have $\forall \gamma \forall^\infty i(n_i(x) = n_i(\gamma \cdot x))$.

Now define a mean $\Phi : \ell^\infty(\Gamma) \to \mathbb{C}$ by

$$\Phi(F) = \int \Phi_{\mathbb{N}}(i \mapsto F(f_{n_i(x)}(x)))d\mu(x),$$

where $\Phi_{\mathbb{N}} : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ is a shift invariant mean as in the proof of 26.6. We will verify that Φ is conjugacy invariant. Denote by $\gamma \cdot F$ the conjugacy action of Γ on $\ell^\infty(\Gamma) : \gamma \cdot F(\delta) = F(\gamma^{-1}\delta\gamma)$. So $\Phi(\gamma^{-1} \cdot F) = \int \Phi_{\mathbb{N}}(i \mapsto F(\gamma f_{n_i(x)}(x)\gamma^{-1}))d\mu(x)$. For each γ, x, if i is sufficiently large,

$$\gamma f_{n_i(x)}(x)\gamma^{-1} = f_{n_i(x)}(\gamma \cdot x) = f_{n_i(\gamma \cdot x)}(\gamma \cdot x),$$

thus

$$\Phi_{\mathbb{N}}(i \mapsto F(\gamma f_{n_i(x)}(x)\gamma^{-1})) = \Phi_{\mathbb{N}}(i \mapsto F(f_{n_i(\gamma \cdot x)}(\gamma \cdot x))),$$

and so

$$\begin{aligned}\Phi(\gamma^{-1}\cdot F) &= \int \Phi_{\mathbb{N}}(i\mapsto F(f_{n_i(\gamma\cdot x)}(\gamma\cdot x)))d\mu(x)\\ &= \int \Phi_{\mathbb{N}}(i\mapsto F(f_{n_i(x)}(x)))d\mu(x)\\ &= \Phi(F).\end{aligned}$$

Since Γ is not inner amenable, Φ must concentrate on $\{1\}$, i.e., $\Phi(\chi_{\Gamma\setminus\{1\}}) = 0$, so

$$\int \Phi_{\mathbb{N}}(i\mapsto \chi_{\Gamma\setminus\{1\}}(f_{n_i(x)}(x)))d\mu(x) = 0.$$

But for each $i, x, f_{n_i(x)}(x) \neq 1$, so

$$\int \Phi_{\mathbb{N}}(i\mapsto \chi_{\Gamma\setminus\{1\}}(f_{n_i(x)}(x)))d\mu(x) = 1,$$

which is a contradiction. □

Remark. The previous proof also shows that if $\theta : \Gamma \to G$ is a homomorphism from Γ to a countable group G and G is not inner amenable relative to $\theta(\Gamma)$ (i.e., every finitely additive probability measure on G, which is invariant under conjugation by elements of $\theta(\Gamma)$, concentrates on 1), then $\theta \in \mathrm{SMOOTH}(a, G)$, for any $a \in \mathrm{ERG}(\Gamma, X, \mu)$.

Corollary 27.15. *Let Γ, G be countable groups, $a \in \mathrm{FRERG}(\Gamma, X, \mu), b \in \mathrm{FRERG}(G, Y, \nu)$ and let $\alpha_\pi \in Z^1(a, G)$ be an orbit equivalence cocycle arising from an orbit equivalence $\pi : X \to Y$ of a and b (i.e., $\alpha_\pi(\gamma, x)\cdot\pi(x) = \pi(\gamma\cdot x)$). If G is not inner amenable, then $\alpha_\pi \in \mathrm{SMOOTH}(a, G)$.*

Proof. We can identify $Z^1(a, G)$ with $Z^1(E_a, G)$ and $Z^1(b, G)$ with $Z^1(E_b, G)$. Since π is an isomorphism of E_a with E_b it induces a homeomorphism π^* between $Z^1(E_a, G)$ and $Z^1(E_b, G)$, preserving the $\tilde{G}$-actions, defined by

$$\pi^*(\alpha)(x, y) = \alpha(\pi^{-1}(x), \pi^{-1}(y)).$$

Then $\pi^*(\alpha_\pi) = \alpha_0$, where $\alpha_0 \in Z^1(b, G)$ is given by $\alpha_0(g, x) = g$. Since $\alpha_0 \in \mathrm{SMOOTH}(b, G)$, clearly $\alpha_\pi \in \mathrm{SMOOTH}(a, G)$. □

(E) We will next discuss some other examples of behavior of the dichotomy ROUGH/SMOOTH.

First let G be a Polish group, E, F be equivalence relations on the spaces $(X, \mu), (Y, \nu)$, resp., and consider the product $E \times F$ on $(X \times Y, \mu \times \nu)$. Then we have a canonical lift map

$$\alpha \in Z^1(E, G) \mapsto \hat{\alpha} \in Z^1(E \times F, G),$$

given by $\hat{\alpha}((x_1, y_1)(x_2, y_2)) = \alpha(x_1, x_2)$. Clearly it is continuous. Also if $g_n \in L(X, \mu, G)$ is such that $g_n \cdot \alpha \to \beta$, then, letting $\hat{g}_n \in L(X \times Y, \mu \times \nu, G)$ be defined by $\hat{g}_n(x, y) = g_n(x)$, we obviously have $\hat{g}_n \cdot \hat{\alpha} \to \hat{\beta}$. Conversely, if $f_n \in L(X \times Y, \mu \times \nu, G)$ and $f_n \cdot \hat{\alpha} \to \hat{\beta}$ pointwise a.e., then, letting $f_n^y(x) = f_n(x, y)$, we have $f_n^y \cdot \alpha \to \beta$ pointwise a.e., ν-a.e.(y).

Indeed, dropping null sets, we can assume that α, β are strict cocycles and f_n is defined everywhere. Let then $A \subseteq X \times Y$ be Borel $(E \times F)$-invariant with $\mu \times \nu(A) = 1$, and such that for $(x_i, y_i) \in A, x_1 E x_2, y_1 F y_2$, we have

$$f_n(x_2, y_2)\alpha(x_1, x_2) f_n(x_1, y_1)^{-1} \to \beta(x_1, x_2).$$

Then, by Fubini, for almost all y we have that $(x, y) \in A$, a.e.(x). For such y we will show that $f_n^y \cdot \alpha \to \beta$ pointwise, a.e. Indeed, fix a Borel E-invariant set $B \subseteq X$ with $\mu(B) = 1$ such that $\forall x \in B (x, y) \in A$. Then if $x_1 E x_2, x_1, x_2 \in B$, we have $(x_1, y), (x_2, y) \in A$ and $(x_1, y) E \times F (x_2, y)$, so $f_n(x_2, y)\alpha(x_1, x_2) f_n(x_1, y)^{-1} \to \beta(x_1, x_2)$, i.e., $f_n^y \cdot \alpha \to \beta$ pointwise on B.

It follows that:

(i) $\alpha \sim \beta \Leftrightarrow \hat{\alpha} \sim \hat{\beta}$,

(ii) $\alpha \in \overline{[\beta]_\sim} \Leftrightarrow \hat{\alpha} \in \overline{[\hat{\beta}]_\sim}$,

(iii) $L(X, \mu, G)_\alpha = \{1\} \Leftrightarrow L(X \times Y, \mu \times \nu, G)_{\hat{\alpha}} = \{1\}$,

(iv) $L(X, \mu, G)_\alpha = \{1\}$ & $\alpha \in \text{SMOOTH}(E, G) \Leftrightarrow L(X \times Y, \mu \times \nu, G)_{\hat{\alpha}} = \{1\}$ & $\hat{\alpha} \in \text{SMOOTH}(E \times F, G)$

(For (iv), $\Leftarrow$ follows from (i), (iii) and the continuity of $\alpha \mapsto \hat{\alpha}$; $\Rightarrow$ follows from (iii), Effros' Theorem and Fubini.)

The following fact will be a consequence of results in Section 29:

There is an ergodic equivalence relation E on (X, μ) and a countable group G such that $\text{SMOOTH}(E, G) = Z^1(E, G)$, and there are $2^{\aleph_0}$ many $\alpha \in Z^1(E, G)$ pairwise not cohomologous such that $L(X, \mu, G)_\alpha = \{1\}$ and also $2^{\aleph_0}$ many $\beta \in Z^1(E, G)$ pairwise not cohomologous such that $L(X, \mu, G)_\beta \neq \{1\}$.

(The equivalence relation E is induced by the shift action of the free group with $\aleph_0$ generators F_∞ on 2^{F_∞}. It will be shown in 29.8 that

$$\text{SMOOTH}(E, F_\infty) = Z^1(E, F_\infty).$$

Now there are $2^{\aleph_0}$ many pairwise non-conjugate homomorphisms $\varphi : F_\infty \to F_\infty$ with $\varphi(F_\infty) = F_\infty$ and thus by 25.4, (ii) the stabilizer of $\varphi \in Z^1(E, F_\infty)$ is equal to $\{1\}$. Also there are $2^{\aleph_0}$ many pairwise non-conjugate homomorphisms $\psi : F_\infty \to F_\infty$ and $h \neq 1$ in F_∞ such that $\psi(F_\infty) \subseteq \{h^n : n \in \mathbb{Z}\}$. Then $h \in L(X, \mu, F_\infty)_\psi \neq \{1\}$.)

Let now $\hat{E} = E \times F$, F ergodic, hyperfinite, which is clearly not E_0-ergodic. Then, by (iii) above and 27.1, for every $\beta \in Z^1(E, G)$ for which $L(X, \mu, G)_\beta \neq \{1\}, \hat{\beta} \in \text{ROUGH}(\hat{E}, G)$ and if $\beta_1 \not\sim \beta_2$, then by (ii) above $\overline{[\hat{\beta}_1]_\sim} \neq \overline{[\hat{\beta}_2]_\sim}$, while if $\alpha \in Z^1(E, G)$ is such that $L(X, \mu, G)_\alpha = \{1\}$, then (iv) above shows that $\hat{\alpha} \in \text{SMOOTH}(\hat{E}, G)$. It follows that for $\hat{E}$ there are $2^{\aleph_0}$ many distinct closures of cohomology classes in $\text{ROUGH}(\hat{E}, G)$ and $2^{\aleph_0}$ many distinct (of course closed) cohomology classes in $\text{SMOOTH}(\hat{E}, G)$.

(F) We next consider another example in which the rough part contains $2^{\aleph_0}$ many closures of cohomology classes.

Let $a_0, a_1, \dots$ be free generators of F_∞, and let $E = E^X_{F_\infty}$ be given by a free, weakly mixing but non E_0-ergodic action of F_∞ on (X, μ). Let $G = F_\infty \times \mathbb{Z}_2$. Let $\mathcal{F} \subseteq p(\mathbb{N})$ be a family of $2^{\aleph_0}$ many almost disjoint infinite subsets of $\mathbb{N}$ (i.e., $X \neq Y \in \mathcal{F} \Rightarrow X \cap Y$ is finite). For $X \in \mathcal{F}$ define the homomorphism $\varphi_X : F_\infty \to G$ by $\varphi_X(a_n) = (a_n, 0)$, if $n \in X$; $= (1, 0)$, if $n \notin X$. First we claim that $\varphi_X \in \mathrm{ROUGH}(E, G)$. Indeed, this follows from 25.4, (ii), and 27.1 as $C_G(\varphi_X(F_\infty)) \supseteq \mathbb{Z}_2 \neq \{1\}$.

Finally, we claim that if $X \neq Y, \varphi_X \notin \overline{[\varphi_Y]_\sim}$. Assume not, towards a contradiction, and find $f_n \in L(X, \mu, G)$ with $f_n \cdot \varphi_Y \to \varphi_X$ pointwise a.e. Thus

$$(*) \qquad \forall\gamma\forall^\infty n(f_n(\gamma \cdot x)\varphi_Y(\gamma)f_n(x)^{-1} = \varphi_X(\gamma)), \text{ a.e.}(x).$$

Define then a mean $\Phi : \ell^\infty(G) \to \mathbb{C}$ by

$$\Phi(F) = \int \Phi_{\mathbb{N}}(\{F(f_n(x))\})d\mu(x),$$

where we use the notation and conventions of the proof of 26.6. If F_∞ acts on G by: $\gamma \cdot g = \varphi_X(\gamma)g\varphi_Y(\gamma)^{-1}$, and thus on $\ell^\infty(G)$ via $\gamma \cdot F(g) = F(\gamma^{-1} \cdot g)$, it is easy to check, using (*), that Φ is invariant under this action. If $\rho(A) = \Phi(\chi_A)$, is the associated finitely additive probability measure on the power set of G, then ρ is invariant under the map $A \mapsto \varphi_X(\gamma)A\varphi_Y(\gamma)^{-1}, \forall\gamma \in \Gamma$. If now $n \in X \setminus Y$, then $\varphi_X(a_n) = (a_n, 0), \varphi_Y(a_n) = (1, 0)$, so $\varphi_X(a_n)A\varphi_Y(a_n)^{-1} = (a_n, 0)A$. Then if $H = \langle a_n : n \in X \setminus Y \rangle \leq F_\infty$, so that $H \times \{0\} \leq G$, then clearly ρ is $H \times \{0\}$-invariant under left-translation, so H is amenable, a contradiction.

(G) We will conclude this section with some open problems. *Below we will assume that the equivalence relation E or the action $a \in A(\Gamma, X, \mu)$ is ergodic but not E_0-ergodic and G is a non-trivial countable group.*

We have seen in Section 25 that $\mathrm{ROUGH}(E, G)$ and $\mathrm{SMOOTH}(E, G)$ are Borel sets. What can be said about their Borel complexity?

We note here that by 27.1, $\alpha \in \mathrm{SMOOTH}(E, G)$ iff $\tilde{G}_\alpha = \{1\}$ and $\tilde{G} \cdot \alpha$ is G_δ iff (by Effros' Theorem) $\tilde{G}_\alpha = \{1\}$ and $\forall\epsilon\exists\delta\forall n(\tilde{d}^1(f_n \cdot \alpha, \alpha) < \delta \Rightarrow \tilde{d}(f_n, 1) < \epsilon)$, where $\{f_n\}$ is dense in $\tilde{G}, d$ is the discrete metric on G and ϵ, δ vary over positive rationals. Therefore $\mathrm{SMOOTH}(E, G)$ is the intersection of the Borel set $\{\alpha \in Z^1(E, G) : \tilde{G}_\alpha = \{1\}\}$ and a $\mathbf{\Pi}^0_3$ set.

Next we have questions concerning category. Can $\mathrm{SMOOTH}(E, G)$ be non-meager? Can there be a cohomology class in $\mathrm{SMOOTH}(E, G)$ which is non-meager and thus clopen? Can the set of cocycles in $Z^1(a, G)$ that are cohomologous to homomorphisms be non-meager? Can the set of reduction cocycles in a given $\overline{[\alpha]_\sim} \subseteq \mathrm{ROUGH}(E, G)$ be non-meager in $\overline{[\alpha]_\sim}$? Can the set of reduction cocycles be non-meager in $Z^1(E, G)$?

Concerning the size of $\mathrm{ROUGH}(E, G), \mathrm{SMOOTH}(E, G)$, let $\mathbf{r}$, resp., $\mathbf{s}$, be the cardinality of the set of cohomology class closures contained in the set $\mathrm{ROUGH}(E, G)$, resp., $\mathrm{SMOOTH}(E, G)$. Thus always $\mathbf{r} \geq 1$. We have seen examples, where $\mathbf{r} = 1, \mathbf{s} = 0; \mathbf{r} = 2^{\aleph_0}, \mathbf{s} = 2^{\aleph_0}$. What possible pairs $(\mathbf{r}, \mathbf{s})$

are realized by appropriately choosing E, G? For instance, is it possible for $\mathbf{r}, \mathbf{s}$ to be finite or countable and $\mathbf{r} > 1$?

We have seen that every cohomology class closure contained in the set $\text{ROUGH}(E, G)$ is path connected. How about $\text{ROUGH}(E, G)$ and $Z^1(E, G)$? Note here that if $1 < \mathbf{r} \leq \aleph_0$, then $\text{ROUGH}(E, G)$ cannot be path connected, since by Sierpinski's Theorem (see van Mill [V], Ex. §4.6, 1) one cannot write the interval [0,1] as a non-trivial disjoint union of countably many closed sets.

Finally, what can be said about the class of groups Γ without property (T) such that for every $a \in \text{ERG}(\Gamma, X, \mu) \setminus \text{E}_0\text{RG}(\Gamma, X, \mu)$ and countable, non-trivial G, we have $Z^1(a, G) = \text{ROUGH}(a, G)$? By 26.4 it contains all amenable groups and by 27.14 it is contained in the class of inner amenable groups.

28. The minimal condition on centralizers

(A) We discuss here a class of groups that will play an important role in the next section.

Let G be a group. For any $A \subseteq G$ we denote by $C_G(A) = \{g \in G : \forall a \in A(ga = ag)\}$ the centralizer of A in G. A *centralizer* in G is a subgroup of the form $C_G(A)$ for some $A \subseteq G$, which we can clearly always take to be a subgroup of G.

A group G satisfies the *minimal condition on centralizers* if there is no strict infinite descending chain $C_0 > C_1 > \dots$ of centralizers (under inclusion). The class of these groups is denoted by $\mathcal{M}_C$.

The following fact gives some equivalent reformulations of this notion.

Proposition 28.1. *Let G be a countable group. Then the following are equivalent:*

(i) G satisfies the minimal condition on centralizers,

(ii) For every increasing sequence $G_0 \leq G_1 \leq \dots$ of subgroups of G,

$$C_G(G_0) \geq C_G(G_1) \geq \dots$$

eventually stabilizes, i.e., $\exists n \forall i \geq n (C_G(G_i) = C_G(G_n))$,

(iii) Same as (ii) with each G_i finitely generated,

(iv) The partial order $H_1 < H_2$ of strict inclusion on the set of centralizers is well founded, i.e., for every nonempty set $\mathcal{C}$ of centralizers there is a minimal element $H_0 \in \mathcal{C}$, i.e., H_0 is such that for no $H_1 \in \mathcal{C}$ we have $H_1 < H_0$,

(v) G satisfies the maximal condition on centralizers, i.e., there is no strict infinite ascending chain $C_0 < C_1 < \dots$ of centralizers,

(vi) For any decreasing sequence $G_0 \geq G_1 \geq \dots$ of subgroups of G,

$$C_G(G_0) \leq C_G(G_1) \leq \dots$$

eventually stabilizes,

(vii) For every subset $A \subseteq G$, there is finite $B \subseteq A$ with $C_G(A) = C_G(B)$.

Proof. The equivalence of (i), (ii) is clear since if $C_0 \geq C_1 \geq \dots$ is a decreasing sequence of centralizers, then $C_i = C_G(H_i)$ for some subgroup $H_i \leq G$ and by letting $G_i = \langle H_0, \dots, H_i \rangle$ we have $G_1 \leq G_2 \leq \dots$ and $C_G(G_i) = C_i$.

Clearly (i) $\Leftrightarrow$ (iv). We next verify that (iv) $\Rightarrow$ (vii). Given $A \subseteq G$, let $\mathcal{C} = \{C_G(B) : B \subseteq A, B \text{ finite}\}$ and let $C_G(B_0)$ be a minimal element of $\mathcal{C}$. We claim that $C_G(B_0) = C_G(A)$. Otherwise there is $g \in C_G(B_0) \setminus C_G(A)$, so that for some $a_0 \in A, ga_0 \neq a_0 g$. Then $g \in C_G(B_0) \setminus C_G(B_0 \cup \{a_0\})$, so $C_G(B_0 \cup \{a_0\}) < C_G(B_0)$, a contradiction. To see that (vii) $\Rightarrow$ (ii), let $G_0 \leq G_1 \leq \dots$ and put $G_\infty = \bigcup_n G_n$. Then $C_G(G_\infty) = C_G(A)$ for some finite $A \subseteq G_\infty$ and thus $A \subseteq G_n$ for some n and $C_G(G_\infty) = C_G(G_n) = C_G(G_i), \forall i \geq n$. Obviously (ii) $\Rightarrow$ (iii). Assume now (iii). This implies that the strict inclusion order $H_1 < H_2$ is well founded on the set of centralizers of *finite* subsets of G. Since this is all we used in the proof of (iv) $\Rightarrow$ (vii), it follows that (iii) $\Rightarrow$ (vii). Thus we have proved the equivalence of (i), (ii), (iii), (iv), (viii).

Below write $C(A) = C_G(A)$ for simplicity. Then note that $A \subseteq B \Rightarrow C(A) \supseteq C(B)$ and $A \subseteq CC(A)$, so $C(A) \supseteq CCC(A)$. If $g \in C(A)$, then g commutes with $CC(A)$, so $g \in CCC(A)$, therefore $C(A) \subseteq CCC(A)$, and thus $C(A) = CCC(A)$, i.e., if $B = C(A)$ is a centralizer, then $B = CC(B)$. So the map $B \mapsto C(B)$ is an inclusion reversing involution on the set of centralizers, which proves the equivalence of (i) with (v) and (v) with (vi), and the proof is complete. □

The class of groups $\mathcal{M}_C$ is quite extensive. It contains all abelian, linear, finitely generated abelian-by-nilpotent groups and is closed under subgroups, finite products and finite extensions; see, e.g., Bryant [Br]. In particular the free groups F_n $(1 \leq n \leq \infty)$ are in $\mathcal{M}_C$.

Perhaps the easiest examples of groups that fail to satisfy the minimal condition on centralizers are infinite direct sums $H_0 \oplus H_1 \oplus \dots$ of centerless groups H_n.

(B) We will next establish a dynamical characterization of these groups that will be the key to our application of this concept in the next section.

Let G be a countable group and consider the (diagonal) action of G on $G^{\mathbb{N}}$ by conjugation

$$g \cdot \{g_n\} = \{g g_n g^{-1}\}.$$

Denote by $F_c(G)$ the corresponding equivalence relation

$$\{g_n\} F_c(G) \{h_n\} \Leftrightarrow \exists g \forall n (g g_n g^{-1} = h_n).$$

Then we have the following result.

Theorem 28.2 (Kechris). *Let G be a countable group. Then the following are equivalent:*

(i) G satisfies the minimal condition on centralizers.

(ii) The equivalence relation $F_c(G)$ is smooth.

Proof. By 22.3, the equivalence relation $F_c(G)$ is not smooth iff there is $\{h_n^0\} \in G^{\mathbb{N}}$ and $\{g_i\} \in G^{\mathbb{N}}$ such that $g_i \cdot \{h_n^0\} \to \{h_n^0\}$ (in $G^{\mathbb{N}}$ with the product topology, G being discrete) but $g_i \cdot \{h_n^0\} \neq \{h_n^0\}, \forall i$.

So assume $F_c(G)$ is not smooth and let $\{g_i\}, \{h_n^0\}$ be as above. Then $\forall n \forall^\infty i (g_i h_n^0 g_i^{-1} = h_n^0)$ but $\forall i \exists n (g_i h_n^0 g_i^{-1} \neq h_n^0)$. Let $A_n = \{h_0^0, h_1^0, \ldots, h_n^0\}$. Then $C_G(A_0) \geq C_G(A_1) \geq \ldots$ does not stabilize, since $\forall n \forall^\infty i (g_i \in C_G(A_n))$ but $\forall i (g_i \notin C_G(\bigcup_n A_n))$.

Conversely, assume $G \notin \mathcal{M}_C$ and let $G_0 \leq G_1 \leq \ldots$ be finitely generated subgroups of G with $C_G(G_0) > C_G(G_1) > \cdots$. Say $G_i = \langle h_0^0, \ldots, h_{n_i}^0 \rangle$, with $n_0 < n_1 < n_2 < \ldots$ Consider then the sequence $\{h_n^0\} \in G^{\mathbb{N}}$ and let $g_i \in C_G(G_i) \setminus C_G(G_{i+1})$. Then $g_i \cdot \{h_n^0\}|n_i = \{h_n^0\}|n_i$, so $g_i \cdot \{h_n^0\} \to \{h_n^0\}$. Also $\forall i \exists j \leq n_{i+1} (g_i h_j^0 g_i^{-1} \neq h_j^0)$, thus $\forall i (g_i \cdot \{h_n^0\} \neq \{h_n^0\})$, so $F_c(G)$ is not smooth. ☐

29. Cohomology IV: The E_0-ergodic case

(A) First let us recall Proposition 26.8:

Proposition 29.1. *Let E be an E_0-ergodic equivalence relation and G a countable group. Then $B^1(E,G) \subseteq \mathrm{SMOOTH}(E,G)$. Similarly for actions.*

Combining this and 27.2 we have the following characterization.

Theorem 29.2. *Let E be an ergodic equivalence relation. Then the following are equivalent:*

(i) E is E_0-ergodic.

(ii) For all countable groups $G, B^1(E,G) \subseteq \mathrm{SMOOTH}(E,G)$.

(iii) For some countable group $G \neq \{1\}, B^1(E,G) \subseteq \mathrm{SMOOTH}(E,G)$.

Similarly for actions.

Schmidt [Sc5], 3.3 has proved a similar result for abelian, locally compact Polish groups G.

(B) When G is abelian, $B^1(E,G)$ is a subgroup of the abelian Polish group $Z^1(E,G)$, so if $B^1(E,G)$ is closed, the equivalence relation $\sim$ is closed and so $\mathrm{SMOOTH}(E,G) = Z^1(E,G)$. As we will see later in this section this is not always true if G is not abelian, so we will next investigate under what circumstances one can still derive that $\mathrm{SMOOTH}(E,G) = Z^1(E,G)$.

Since G can be viewed as a closed subgroup of $L(X,\mu,G) = \tilde{G}$, by identifying $g \in G$ with the constant function $x \mapsto g$, the action of $\tilde{G}$ on $Z^1(E,G)$ restricts to the action of G on $Z^1(E,G)$ given by

$$g \cdot \alpha(x,y) = g\alpha(x,y)g^{-1},$$

i.e., conjugation. We will denote by $F_c(E,G)$ the corresponding equivalence relation, which is a subequivalence relation of $\sim$. Similarly we define $F_c(a,G)$ for $a \in A(\Gamma,X,\mu)$. Recall from Section 28, **(B)** that $F_c(G)$ denotes the equivalence relation of conjugacy on $G^{\mathbb{N}}$ and that for equivalence relations E, F on X, Y, resp., we let

$$E \leq_B F,$$

if E can be *Borel reduced* to F, i.e., if there is Borel $f : X \to Y$ with

$$xEy \Leftrightarrow f(x)Ff(y).$$

Proposition 29.3. *For any ergodic equivalence relation E on (X, μ) and countable group G,*

$$F_c(G) \leq_B F_c(E, G)|B^1(E, G).$$

Similarly for actions.

Proof. Fix a partition $X = \bigsqcup_n X_n, n \geq 0$, where $\mu(X_n) = 2^{-n-1}$. Given $\{g_n\} \in G^{\mathbb{N}}$, let $f_{\{g_n\}} : X \to G$ be defined by

$$f_{\{g_n\}}(x) = \begin{cases} 1, \text{ if } x \in X_0, \\ g_{n-1}, \text{ if } x \in X_n, n > 0. \end{cases}$$

Put $\alpha_{\{g_n\}}(x, y) = f_{\{g_n\}}(y) f_{\{g_n\}}(x)^{-1} \in B^1(E, G)$. We claim that

$$\{g_n\} F_c(G) \{h_n\} \Leftrightarrow \alpha_{\{g_n\}} F_c(E, G) \alpha_{\{h_n\}}.$$

$\Rightarrow$: Fix g such that $g g_n g^{-1} = h_n, \forall n$. Then $g\alpha_{\{g_n\}}(x, y) g^{-1} = \alpha_{\{h_n\}}(x, y)$.

$\Leftarrow$: Let g be such that $g\alpha_{\{g_n\}}(x, y) g^{-1} = \alpha_{\{h_n\}}(x, y)$, i.e.,

$$g f_{\{g_n\}}(y) f_{\{g_n\}}(x)^{-1} g^{-1} = f_{\{h_n\}}(y) f_{\{h_n\}}(x)^{-1},$$

for xEy. By ergodicity X_n meets (almost) every E-class, so there are xEy such that $x \in X_0, y \in X_{n+1}$. Then $g g_n g^{-1} = h_n$. □

I do not know if, conversely, $F_c(E, G)|B^1(E, G) \leq_B F_c(G)$. However we have the following.

Proposition 29.4. *For any ergodic equivalence relation E on (X, μ) and countable group G, if $F_c(G)$ is smooth, then $F_c(E, G)$ is smooth. Similarly for actions.*

Proof. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a countable group acting in a Borel way on X so that $E = E_\Gamma^X$.

Assume that $F_c(G)$ is smooth. By 22.3 it is enough to show that if $\{g_n\} \in G^{\mathbb{N}}, \alpha \in Z^1(E, G)$ and $g_n \cdot \alpha \to \alpha$, then for some $n, g_n \cdot \alpha = \alpha$. By going to a subsequence, we can assume that

$$\forall \gamma \forall^\infty n (g_n \alpha(x, \gamma \cdot x) g_n^{-1} = \alpha(x, \gamma \cdot x)), \text{ a.e.}(x).$$

Define $\varphi : X \to G^{\mathbb{N}}$ by

$$\varphi(x) = \{\alpha(x, \gamma_k \cdot x)\}_{k=0}^{\infty}.$$

Then for the conjugacy action of G on $G^{\mathbb{N}}$ we have

$$g_n \cdot \varphi(x) \to \varphi(x), \quad \text{a.e.}(x).$$

Since $F_c(G)$ is smooth, by 22.3 again, we have that $\forall^\infty n (g_n \cdot \varphi(x) = \varphi(x))$, i.e.,

$$\exists N \forall n \geq N \forall \gamma (g_n \alpha(x, \gamma \cdot x) g_n^{-1} = \alpha(x, \gamma \cdot x)), \text{ a.e.}(x).$$

Thus there exists N_0 and a set of positive measure $A_0 \subseteq X$ (which is therefore an a.e. complete section of E) such that

$$x \in A_0 \Rightarrow \forall n \geq N_0 \forall \gamma (g_n \alpha(x, \gamma \cdot x) g_n^{-1} = \alpha(x, \gamma \cdot x)).$$

We now claim that

$$\forall n \geq N_0 \forall \gamma (g_n \alpha(x, \gamma \cdot x) g_n^{-1} = \alpha(x, \gamma \cdot x)), \text{ a.e.}(x),$$

therefore $\forall^\infty n(g_n \cdot \alpha = \alpha)$. Indeed, by discarding a null set, we can assume that for any x there is $\delta \in \Gamma$ such that $\delta^{-1} \cdot x \in A_0$. So fix $x, \gamma, n \geq N_0$ and let $\delta^{-1} \cdot x = y \in A_0$. Then letting $\alpha^*(\gamma, x) = \alpha(x, \gamma \cdot x)$, we have

$$\alpha^*(\gamma, \delta \cdot y) = \alpha^*(\gamma\delta, y)\alpha^*(\delta, y)^{-1}.$$

But $g_n \alpha^*(\gamma\delta, y) g_n^{-1} = \alpha^*(\gamma\delta, y)$ and $g_n \alpha^*(\delta, y) g_n^{-1} = \alpha^*(\delta, y)$, so

$$\begin{aligned} g_n \alpha^*(\gamma, x) g_n^{-1} &= (g_n \alpha^*(\gamma\delta, y) g_n^{-1})(g_n \alpha^*(\delta, y)^{-1} g_n^{-1}) \\ &= \alpha^*(\gamma\delta, y)\alpha^*(\delta, y)^{-1} = \alpha^*(\gamma, \delta \cdot y) = \alpha^*(\gamma, x). \end{aligned}$$

□

Corollary 29.5. *For any countable group G, the following are equivalent:*

(i) G satisfies the minimal condition on centralizers.

(ii) $F_c(G)$ is smooth.

(iii) For every ergodic equivalence relation $E, F_c(E, G)$ is smooth.

(iv) For some ergodic equivalence relation $E, F_c(E, G)$ is smooth.

We now have:

Theorem 29.6 (Kechris). *Let E be an E_0-ergodic equivalence relation and assume G is countable and satisfies the minimal condition on centralizers. Then* $\mathrm{SMOOTH}(E, G) = Z^1(E, G)$. *Similarly for actions.*

Proof. Fix $\alpha \in Z^1(E, G)$ and consider its stabilizer $\tilde{G}_\alpha$ for the action of $L(X, \mu, G) = \tilde{G}$ on $Z^1(E, G)$. By Effros' Theorem it is enough to show that if $g_n \in \tilde{G}$ and $g_n \cdot \alpha \to \alpha$, then $g_n \tilde{G}_\alpha \to \tilde{G}_\alpha$ (in $\tilde{G}/\tilde{G}_\alpha$). By passing to subsequences we can assume that on a measure 1 set in X we have

$$(*) \qquad xEy \Rightarrow \forall^\infty n(g_n(y)\alpha(x, y) g_n(x)^{-1} = \alpha(x, y)).$$

Define the equivalence relation $F_{c,0}(G)$ on $G^{\mathbb{N}}$ by

$$\{f_n\} F_{c,0}(G) \{h_n\} \Leftrightarrow \exists g \forall^\infty n (g f_n g^{-1} = h_n).$$

Let also F_k on $G^{\mathbb{N}}$ be defined by

$$\{f_n\} F_k \{h_n\} \Leftrightarrow \exists g \forall n \geq k (g f_n g^{-1} = h_n).$$

Then $F_0 \subseteq F_1 \subseteq \ldots, F_{c,0}(G) = \bigcup_k F_k$ and clearly F_k is Borel reducible to $F_c(G)$, which is smooth, thus F_k is smooth and $F_{c,0}(G)$ is hyperfinite.

Now we have by (*)

$$xEy \Rightarrow \{g_n(x)\} F_{c,0}(G) \{g_n(y)\},$$

so, since E is E_0-ergodic, there is $\{h_n^0\} \in G^{\mathbb{N}}$ such that

$$\{g_n(x)\}F_{c,0}(G)\{h_n^0\}, \text{ a.e.}(x),$$

i.e. $\exists g \forall^\infty n(g_n(x) = gh_n^0 g^{-1})$, a.e.$(x)$. So there is $g \in \tilde{G}$ such that

$$(**) \qquad \forall^\infty n(g_n(x) = g(x)h_n^0 g(x)^{-1}), \text{ a.e.}(x).$$

Put $g^{-1} \cdot \alpha = \beta$, i.e., $\beta(x, y) = g(y)^{-1}\alpha(x, y)g(x)$. Then from $(^*), (^{**})$ we have, a.e.,

$$xEy \Rightarrow \forall^\infty n(h_n^0 \beta(x, y)(h_n^0)^{-1} = \beta(x, y)),$$

so $h_n^0 \cdot \beta \to \beta$. Since, by 29.4, $F_c(E, G)$ is smooth, by 22.3 we have $\forall^\infty n(h_n^0 \cdot \beta = \beta)$ or $\forall^\infty n(gh_n^0 g^{-1} \in \tilde{G}_\alpha)$. Let $h_n = gh_n^0 g^{-1} \in \tilde{G}_\alpha$. Then by $(^{**})$

$$\forall^\infty n(g_n(x) = h_n(x)), \text{a.e.}(x),$$

so $d(g_n(x), h_n(x)) \to 0$, where d is the discrete metric on G, and then by Lebesgue Dominated Convergence $\tilde{d}(g_n, h_n) \to 0$, i.e., $\tilde{d}(g_n, \tilde{G}_\alpha) \to 0$, so $g_n\tilde{G}_\alpha \to \tilde{G}_\alpha$ in $\tilde{G}/\tilde{G}_\alpha$. □

Combining 27.2 and 29.6 we see that if E is an ergodic equivalence relation and G is a countable group, then we have:

(i) If E is not E_0-ergodic, then SMOOTH$(E, G) \neq Z^1(E, G)$ and $\sim$ on $Z^1(E, G)$ is not smooth.

(ii) If E is E_0-ergodic and $G \in \mathcal{M}_C$, then SMOOTH$(E, G) = Z^1(E, G)$ and $\sim$ is smooth.

Thus in particular we have:

Corollary 29.7. *Let G be a countable group satisfying the minimal condition on centralizers. Then for any ergodic E the following are equivalent:*

(i) E is E_0-ergodic.

(ii) SMOOTH$(E, G) = Z^1(E, G)$.

Similarly for actions.

There are examples of E_0-ergodic E in which $B^1(E, G) = Z^1(E, G)$, so obviously SMOOTH$(E, G) = Z^1(E, G)$, but $G \notin \mathcal{M}_C$. For example, let E be given by a free ergodic action of a property (T) group and let $G = G_0 \oplus G_1 \oplus \dots$, where G_i are centerless, torsion free and amenable. However in certain situations the smoothness of $\sim$ on $Z^1(E, G)$ implies that $G \in \mathcal{M}_C$.

Theorem 29.8 (Kechris). *Let a be an E_0-ergodic, weak mixing action of F_∞. Then for any countable G the following are equivalent:*

(i) G satisfies the minimal condition on centralizers.

(ii) SMOOTH$(a, G) = Z^1(a, G)$.

Proof. We have already seen that (i) $\Rightarrow$ (ii). Assume now that (i) fails. Then $F_c(G)$ is not smooth. But notice that $F_c(G)$ is really the same as the conjugacy relation on Hom(F_∞, G), which by 20.1 can be Borel reduced to $\sim$

on $Z^1(a, G)$. It follows that the latter is not smooth, i.e., $\text{SMOOTH}(a, G) \neq Z^1(a, G)$. □

I do not know if 29.8 holds as well for $F_n, 2 \leq n < \infty$. It is also unclear what is the possible complexity of the equivalence relation $\sim$ on $Z^1(E, G)$, when E is E_0-ergodic and $G \notin \mathcal{M}_C$.

(C) We will now see some implications of the previous results to the general problem discussed in Jones-Schmidt [JS] of understanding when for an action $a \in \text{FRERG}(\Gamma, X, \mu)$ the outer automorphism group $\text{Out}(E_a)$ is Polish or equivalently $[E_a]$ is closed in $N[E_a]$. We have already seen various results in that direction, e.g., this holds for *any* $a \in \text{FR}(\Gamma, X, \mu)$ if Γ is not inner amenable (see 9.1). It also holds if $a \in \text{ERG}(\Gamma, X, \mu)$ and $C_\mu(E_a) > 1$, even if the action is not free (see 8.1). Jones-Schmidt [JS], 4.5 raise the question of whether there is a dynamical characterization of the closedness of $[E_a]$ in $N[E_a]$. They point out that there appears to be no direct connection between the E_0-ergodicity of a and the closedness of $[E_a]$ in $N[E_a]$. It is easy to see that $[E_a]$ being closed in $N[E_a]$ does not imply E_0-ergodicity. (For example, take Γ to be a group that is not inner amenable and does not have property (T) and take $a \in \text{FRERG}(\Gamma, X, \mu) \setminus \text{E}_0\text{RG}(\Gamma, X, \mu)$.) Then they refer to the paper Connes-Jones [CJ1] for an example of a group Γ and a free action $a \in \text{E}_0\text{RG}(\Gamma, X, \mu)$ such that $[E_a]$ is not closed in $N[E_a]$. We present below a variation of this example.

Example (Connes-Jones [CJ1]). Let Γ_0 be a weakly commutative ICC group and fix $\{\gamma_n\} \subseteq \Gamma_0 \setminus \{1\}$ such that $\forall \gamma \in \Gamma_0 \forall^\infty n(\gamma\gamma_n = \gamma_n\gamma)$. Consider an ergodic, free action b of Γ_0 on (X, μ) which is such that $\gamma_n^b \xrightarrow{w} 1$, and $[E_b]$ is not closed in $N[E_b]$. For example, we can take this action to be the conjugacy shift action of Γ_0 on $X = 2^{\Gamma_0^*}$ (see the proof of 9.5). Let Δ_0 be a non-amenable group and let $\Gamma = \Gamma_0 \times \Delta_0$ act on $Y = X^{\Delta_0}$, by letting Δ_0 acting by shift and Γ_0 acting coordinatewise via b. Call this action a. Since the Δ_0-action is E_0-ergodic, the $(\Gamma_0 \times \Delta_0)$-action is E_0-ergodic. One can also easily see that it is free: If $(\gamma_0, \delta_0) \cdot p = p$, for a positive measure set of $p \in X^{\Delta_0}$, then $\forall \delta \in \Delta_0(\gamma_0 \cdot p(\delta_0^{-1}\delta) = p(\delta))$. If $\delta_0 \neq 1$, this clearly holds on a null set of $p \in X^{\Delta_0}$. So assume $\delta_0 = 1$. Thus $\forall \delta \in \Delta_0(\gamma_0 \cdot p(\delta) = p(\delta))$. But if $\gamma_0 \neq 1, \{x : \gamma_0 \cdot x = x\}$ is null, so again this only holds on a null set. Thus $\gamma_0 = 1$.

We will finally show that $[E_a]$ is not closed in $N[E_a]$, by showing that $\gamma_n^a \to 1$ in $N[E_a]$ but clearly $\gamma_n^a \not\to 1$ uniformly. Since $\{\gamma_n\}$ witnesses the weak commutativity of Γ_0 and thus of Γ it is enough, by 6.2, to show that $\gamma_n^a \xrightarrow{w} 1$. Again for that it is enough to show that if $A_1, \dots, A_k \subseteq X$ are Borel sets and μ^k is the product measure on X^k, then

$$\mu^k((\gamma_n^b(A_1) \times \cdots \times \gamma_n^b(A_k))\Delta(A_1 \times \cdots \times A_k)) \to 0.$$

As $\mu(\gamma_n^b(A_i)\Delta A_i) \to 0, \forall i \leq k$, this is clear.

This example shows that there is a weakly commutative ICC group Γ and a free, E_0-ergodic action a of Γ on (X, μ) with $[E_a]$ not closed in $N[E_a]$. Note that centerless weakly commutative groups do not have the minimal condition on centralizers. Because if $\{\gamma_n\}$ witnesses the weak commutativity of Γ and Γ is in $\mathcal{M}_C$, then $\Gamma = \bigcup_n C_G(\{\gamma_k\}_{k=n}^\infty))$ and so, by 28.1, $\Gamma = C_G(\{\gamma_k\}_{k=n_0}^\infty)$, for some n_0, i.e., $\gamma_k \in Z(\Gamma)$, if $k \geq n_0$, a contradiction. We will now show that for ICC groups Γ with the minimal condition on centralizers, E_0-ergodicity of an action a implies the closedness of $[E_a]$ in $N[E_a]$. Thus for "nice" groups Γ, there is indeed a direct implication from E_0-ergodicity to the closedness of $[E_a]$ in $N[E_a]$.

For any countable group Γ and action $a \in A(\Gamma, X, \mu)$, we denote by α_0 the cocycle in $Z^1(a, \Gamma)$ given by

$$\alpha_0(\gamma, x) = \gamma.$$

We now have the following result. Keep in mind that an action has no non-trivial *finite factors* (i.e., homomorphisms into actions on non-trivial finite probability spaces) iff every finite index subgroup acts ergodically.

Proposition 29.9. *Let Γ be a countable group and $a \in \mathrm{FRERG}(\Gamma, X, \mu)$. Assume that either Γ is* ICC *or else Γ is centerless and every finite index subgroup of Γ acts ergodically (e.g., a is weak mixing). Then if $\alpha_0 \in \mathrm{SMOOTH}(a, \Gamma)$, the group $[E_a]$ is closed in $N[E_a]$.*

Proof. Let $E = E_a$ and let $T_n \in [E]$ be such that $T_n \to 1$ in $N[E]$ in order to show that $T_n \xrightarrow{u} 1$. Put $f_n(x) = \alpha_0(x, T_n(x))$, so that $T_n(x) = f_n(x) \cdot x$. We have, identifying γ with γ^a,

$$\forall \gamma \in \Gamma(T_n \gamma T_n^{-1} \xrightarrow{u} \gamma),$$

i.e., if $A_n^\gamma = \{x : T_n \gamma T_n^{-1}(x) \neq \gamma \cdot x\}$, then $\mu(A_n^\gamma) \to 0$. Now $\{x : f_n(\gamma \cdot x)\gamma f_n(x)^{-1} \neq \gamma\} \subseteq \{x : T_n(\gamma T_n^{-1}(T_n(x))) \neq \gamma \cdot T_n(x)\} = T_n^{-1}(A_n^\gamma)$, so $\forall \gamma (\mu(\{x : f_n(\gamma \cdot x)\gamma f_n(x)^{-1} \neq \gamma\}) \to 0)$, thus $f_n \cdot \alpha_0 \to \alpha_0$.

Next we note that $\tilde{\Gamma}_{\alpha_0}$ $(= L(X, \mu, \Gamma)_{\alpha_0}) = \{1\}$. Indeed if $f \cdot \alpha_0 = \alpha_0$, i.e., $f(\gamma \cdot x)\gamma f(x)^{-1} = \gamma$, then $f(\gamma \cdot x) = \gamma f(x) \gamma^{-1}$, so $f : X \to \Gamma$ is a homomorphism of the Γ-action on X and the conjugacy action on Γ. Thus $f_*\mu$ concentrates on a finite conjugacy class of Γ. Our hypothesis then implies that $f_*\mu$ concentrates on $\{1\}$, i.e., $f = 1$.

So, by Effros' Theorem, since $\alpha_0 \in \mathrm{SMOOTH}(a, \Gamma)$, we have $f_n \to 1$ in $\tilde{\Gamma}$, i.e., $\mu(\{x : f_n(x) \neq 1\}) = \mu(\{x : T_n(x) \neq x\}) = \delta_u(T_n, 1) \to 1$. □

Theorem 29.10 (Kechris). *Let Γ be a countable group satisfying the minimal condition on centralizers and let $a \in A(\Gamma, X, \mu)$ be a free, E_0-ergodic action of Γ. Assume that either Γ is* ICC *or else Γ is centerless and every finite index subgroup of Γ acts ergodically (e.g., a is weak mixing). Then $[E_a]$ is closed in $N[E_a]$.*

Proof. By 29.6 and 29.9. □

Recall from Section 9 that it is still an open problem to characterize the ICC groups Γ such that for every $a \in \text{FRERG}(\Gamma, X, \mu), [E_a]$ is closed in $N[E_a]$. One can ask the "dual" question of characterizing the groups Γ such that for every $a \in \text{FRERG}(\Gamma, X, \mu), [E_a]$ is *not* closed in $N[E_a]$. We have already seen in 7.2 that every amenable group Γ satisfies this property, in fact $[E_a]$ is dense in $N[E_a]$ in this case. The next result provides the converse, for centerless $\Gamma \in \mathcal{M}_C$.

Theorem 29.11 (Kechris). *Let* Γ *be a centerless countable group with the minimal condition on centralizers. Then the following are equivalent:*

(i) Γ *is amenable,*

(ii) $\forall a \in \text{FRERG}(\Gamma, X, \mu)([E_a]$ *is not closed in* $N[E_a])$*,*

(iii) If s *is the shift action of* Γ *on* 2^Γ*, then* $[E_s]$ *is not closed in* $N[E_s]$*,*

(iv) $\forall a \in \text{FRERG}(\Gamma, X, \mu)([E_a]$ *is dense in* $N[E_a])$*,*

(v) $[E_s]$ *is dense in* $N[E_s]$*.*

Proof. (i) $\Rightarrow$ (ii) - (v) follows from 7.2. If Γ is not amenable, then s is E_0-ergodic and mixing, so, by 29.10, $[E_s]$ is closed in $N[E_s]$, i.e., iii) fails. So (i) - (iii) are equivalent. Moreover, in this case, $[E_s]$ is not dense in $N[E_s]$, else $[E_s] = N[E_s]$, which is absurd as the map $x \in 2^\Gamma \mapsto \overline{x} \in 2^\Gamma$ (where $\overline{x}(\gamma) = 1 - x(\gamma)$) is in $N[E_s] \setminus [E_s]$. Thus (i), (iv), (v) are equivalent. □

Remark. It should be pointed out here that if $E = R \times F$ is the product of two equivalence relations on $(X, \mu), (Y, \nu)$, resp., and $[F]$ is not closed in $N[F]$, then $[E]$ is not closed in $N[E]$. This is clear since if $T_n \in [F], T_n \to 1$ in $N[F]$ but $T_n \not\to 1$ uniformly and we define $T'_n(x, y) = (x, T_n(y))$, then $T'_n \in [E], T_n \to 1$ in $N[E]$ but $T'_n \not\to 1$ uniformly. It follows that if a group Γ has a free ergodic action a with $[E_a]$ not closed in $N[E_a]$, so does any product $\Gamma \times \Delta, \Delta$ a countable group.

There is a somewhat different version of 29.10 that may be worth stating as it seems to provide some insight concerning the role of weak commutativity in the Connes-Jones counterexample. We use below the following variant of a terminology from Jackson-Kechris-Louveau [JKL]: If E is a countable Borel equivalence relation on a standard Borel space X, we call E *measure-hyperfinite* if for every measure μ on X, E is hyperfinite except perhaps on a μ-null set.

Theorem 29.12 (Kechris). *Let* Γ *be a countable group and let* $a \in A(\Gamma, X, \mu)$ *be a free,* E_0*-ergodic action, in which every finite index subgroup acts ergodically* (*e.g.,* a *is weak mixing*)*. Assume that* Γ *is not weakly commutative and* $F_{c,0}(\Gamma)$ *is measure-hyperfinite. Then* $\alpha_0 \in \text{SMOOTH}(a, \Gamma)$ *and thus* $[E_a]$ *is closed in* $N[E_a]$*.*

Proof. As in the proof of 29.9, $\tilde{\Gamma}_{\alpha_0} = \{1\}$. We next show that $\alpha_0 \in \text{SMOOTH}(a, \Gamma)$. As in the proof of 29.6, we start with a sequence $g_n \in \tilde{\Gamma}$ such that,

$$\forall \gamma \forall^\infty n (g_n(\gamma \cdot x) \gamma g_n(x)^{-1} = \gamma), \text{ a.e.}(x),$$

in order to show that $g_n \to 1$. As in that proof, the assumption that $a \in \mathrm{E_0RG}(\Gamma, X, \mu)$ and $F_{c,0}(\Gamma)$ is measure-hyperfinite implies that there is $\{h_n^0\} \in \Gamma^{\mathbb{N}}$ and $g \in \tilde{\Gamma}$ such that

$$\forall^\infty n(g_n(x) = g(x)h_n^0 g(x)^{-1}), \text{ a.e.}(x),$$

and so

$$\forall\gamma\forall^\infty n(g_n(\gamma\cdot x) = g(\gamma\cdot x)h_n^0 g(\gamma\cdot x)^{-1}), \text{ a.e.}(x),$$

thus

$$\forall\gamma\forall^\infty n(g(\gamma\cdot x)^{-1}\gamma g(x) \text{ commutes with } h_n^0), \text{ a.e.}(x).$$

Put

$$H = \{\gamma \in \Gamma : \forall^\infty n(\gamma h_n^0 = h_n^0\gamma)\} \leq \Gamma.$$

So

$$\forall\gamma(g(\gamma\cdot x)^{-1}\gamma g(x) \in H), \text{ a.e.}(x),$$

i.e.,

$$\forall\gamma(g(\gamma\cdot x)H = \gamma g(x)H), \text{ a.e.}(x).$$

Put

$$\varphi(x) = g(x)H,$$

so that $\varphi : X \to \Gamma/H$ is a Borel homomorphism of the Γ-action on X into the Γ-action on Γ/H. Then $\varphi_*\mu$ is a Γ-invariant measure on Γ/H, so $[\Gamma : H] < \infty$. Since the Γ-action on X has no non-trivial finite factors, we must have $\Gamma = H$. Thus since Γ is not weakly commutative, $\forall^\infty n(h_n^0 = 1)$, and so $\forall^\infty n(g_n(x) = 1)$, a.e.$(x)$, and therefore $g_n \to 1$. □

Of course if $\Gamma \in \mathcal{M}_C$, then $F_{c,0}(\Gamma)$ is hyperfinite but there are groups Γ for which $F_{c,0}(\Gamma)$ is measure-hyperfinite, which are not in $\mathcal{M}_C$, e.g., any amenable group not in $\mathcal{M}_C$. Thus the extent of the class of groups Γ for which $F_{c,0}(\Gamma)$ is measure-hyperfinite is not clear. Simon Thomas (private communication) constructed examples of groups Γ for which $F_{c,0}(\Gamma)$ is not measure-hyperfinite. They also satisfy the following additional properties: ICC, inner amenable, not weakly commutative and have cost 1 (and fixed price).

(D) We will conclude this section with some further facts related to the last remark in Section 7 and which also give an alternative proof of (a somewhat weaker version of) 29.9.

Let $a \in \mathrm{FR}(\Gamma, X, \mu), \Gamma$ infinite, and let $T \in N[E_a]$. Then as in Section 21, **(C)** we associate to each $T \in N[E_a]$ the cocycle $\alpha_T \in Z^1(a,\Gamma)$ defined by

$$T(\gamma\cdot x) = \alpha_T(\gamma, x)\cdot T(x).$$

Proposition 29.13. *The map $T \mapsto \alpha_T$ is continuous from $(N[E_a], \tau_{N[E_a]})$ into $Z^1(a,\Gamma)$.*

Proof. Put $E = E_a$. Assume that $T_n \to T$ in $N[E]$ in order to show that $\alpha_{T_n} \to \alpha_T$ in $Z^1(a,\Gamma)$. Note that $T_n \to T$ in $N[E]$ implies that $T_n \stackrel{w}{\to} T$ and so $T_n^{-1} \stackrel{w}{\to} T^{-1}$ and thus $T_n^{-1}T \stackrel{w}{\to} 1$.

Fix $\epsilon > 0, \gamma \in \Gamma$. We will find N such that $\forall n \geq N$,

$$\mu(\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, x)\}) < \epsilon.$$

Put $A_\delta = \{x : \alpha_T(\gamma, x) = \delta\}, A^n_\delta = \{x : \alpha_{T_n}(\gamma, x) = \delta\}$. Then there is a finite $F \subseteq \Gamma$ such that $\sum_{\delta \notin F} \mu(A_\delta) < \epsilon/6$. Now $\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, x)\} = \bigcup_\delta (A_\delta \cap \bigcup_{\delta' \neq \delta} A^n_{\delta'})$. Choose M such that $\forall n \geq M$,

$$\forall \delta \in F(\mu(A_\delta \Delta T_n^{-1} T(A_\delta)) < \frac{\epsilon}{6|F|}).$$

This is possible, since $T_n^{-1} T \xrightarrow{w} 1$. Then

$$\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, x)\} = \bigcup_{\delta \notin F} (A_\delta \cap \bigcup_{\delta' \neq \delta} A^n_{\delta'}) \cup \bigcup_{\delta \in F} (A_\delta \cap \bigcup_{\delta' \neq \delta'} A^n_{\delta'}),$$

and

$$\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, T^{-1} T_n(x))\} = \bigcup_{\delta \notin F} (T_n^{-1} T(A_\delta) \cap \bigcup_{\delta' \neq \delta} A^n_{\delta'}) \cup \bigcup_{\delta \in F} (T_n^{-1} T(A_\delta) \cap \bigcup_{\delta' \neq \delta'} A^n_{\delta'}),$$

so

$$\mu(\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, x)\} \Delta \{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, T^{-1} T_n(x))\})$$

$$< \epsilon/6 + \epsilon/6 + \sum_{\delta \in F} \mu(A_\delta \Delta T_n^{-1} T(A_\delta)) < \epsilon/2\cdot$$

We also have $T_n \gamma T_n^{-1} \to T \gamma T^{-1}$ uniformly, so we can find $N > M$ such that for $n \geq N$,

$$\mu(\{x : T_n \gamma T_n^{-1}(x) \neq T \gamma T^{-1}(x)\}) < \epsilon/2.$$

By the freeness of the action,

$$\{x : \alpha_{T_n}(\gamma, T_n^{-1}(x)) \neq \alpha_T(\gamma, T^{-1}(x))\} \subseteq \{x : T_n \gamma T_n^{-1}(x) \neq T \gamma T^{-1}(x)\},$$

so

$$\mu(\{x : \alpha_{T_n}(\gamma, T_n^{-1}(x)) \neq \alpha_T(\gamma, T^{-1}(x))\}) < \epsilon/2.$$

Therefore

$$\mu(\{x : \alpha_{T_n}(\gamma, T_n^{-1}(x)) \neq \alpha_T(\gamma, T^{-1}(x))\}) = \mu(\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, T^{-1} T_n(x))\}) < \epsilon/2,$$

so we have

$$\mu(\{x : \alpha_{T_n}(\gamma, x) \neq \alpha_T(\gamma, x)\}) < \epsilon.$$

□

Recall that for an action $a \in A(\Gamma, X, \mu), C_a = \{T \in \mathrm{Aut}(X,\mu) : TaT^{-1} = a\}$ is the stabilizer of the action (also denoted by C_Γ if the action is understood). Then we have:

Proposition 29.14 (Furman [Fu]). *For any* $a \in \mathrm{FR}(\Gamma, X, \mu)$, $T \in N[E_a]$,

$$T \in C_a[E_a] \Leftrightarrow \alpha_T \sim \alpha_0.$$

Proof. $\Rightarrow$: If $T = SU$, where $S \in C_a, U \in [E_a]$, and $f : X \to \Gamma$ is such that $f(x) \cdot x = U(x)$, then $U(\gamma \cdot x) = f(\gamma \cdot x)\gamma f(x)^{-1} \cdot U(x)$, so $T(\gamma \cdot x) = S(U(\gamma \cdot x)) = S(f(\gamma \cdot x)\gamma f(x)^{-1} \cdot U(x)) = f(\gamma \cdot x)\gamma f(x)^{-1} \cdot T(x)$, so $\alpha_T(\gamma, x) = f(\gamma \cdot x)\gamma f(x)^{-1}$, i.e., $\alpha_T \sim \alpha_0$.

$\Leftarrow$: Suppose that $\alpha_T(\gamma, x) = f(\gamma \cdot x)\gamma f(x)^{-1}$, for some $f \in L(X, \mu, \Gamma)$. Then $T(\gamma \cdot x) = f(\gamma \cdot x)\gamma f(x)^{-1} \cdot T(x)$. Put $S(x) = f(x)^{-1} \cdot T(x)$, so that $S(\gamma \cdot x) = \gamma \cdot S(x)$. We claim that $S \in \mathrm{Aut}(X, \mu)$. First notice that S maps any orbit $\Gamma \cdot x$ in a 1-1 way onto the orbit $\Gamma \cdot S(x) = \Gamma \cdot T(x)$ and thus, in particular, S is a Borel bijection of X. Next we will verify that S preserves μ, and for that it is enough to show that for each $\delta \in \Gamma$, if $A_\delta = f^{-1}(\{\delta\})$, then $S|A_\delta$ is measure preserving. But for $x \in A_\delta, S(x) = \delta^{-1} \cdot T(x)$, so this is clear.

Thus $S \in C_a$. Put $U(x) = f(S^{-1}(x)) \cdot x$. Then $US = T$, so $U = TS^{-1} \in \mathrm{Aut}(X, \mu)$. Moreover $U \in [E_a]$. So $T \in [E_a]C_a = C_a[E_a]$. □

From 29.13 and 29.14, it follows that if $a \in \mathrm{FR}(\Gamma, X, \mu)$ and the cocycle $\alpha_0 \in \mathrm{SMOOTH}(a, \Gamma)$, then $C_a[E_a]$ is closed in $N[E_a]$, so by the last remark in Section 7, if moreover $C_a \cap [E_a] = \{1\}, [E_a]$ is closed in $N[E_a]$. Recall from 14.6 that if $a \in \mathrm{FRERG}(\Gamma, X, \mu)$, then $C_a \cap [E_a] = \{1\}$ if either Γ is ICC or else Γ is centerless and every infinite subgroup of Γ acts ergodically. Compare this with 29.9.

30. Cohomology V: Actions of property (T) groups

(A) For actions of property (T) groups the structure of the cohomology classes is much simpler in view of the following result.

Theorem 30.1 (Popa [Po3]). *Let* Γ *be a countable group that has property* (T)*, let* G *be a countable group and let* $a \in \mathrm{ERG}(\Gamma, X, \mu)$*. Then every cohomology class in* $Z^1(a, G)$ *is clopen* (*so* $H^1(a, G)$ *is countable*).

Proof (Furman [Fu2]). We will find $\delta > 0$ such that if $\alpha, \beta \in Z^1(a, G)$ and $\tilde{d}^1(\alpha, \beta) < \delta$, then $\alpha \sim \beta$.

Since Γ has property (T) there is a finite symmetric $F \subseteq \Gamma$ and $\epsilon > 0$ such that for any unitary representation $\pi : \Gamma \to U(H)$, which admits a unit vector v_0 with $\forall \gamma \in F(|\langle \pi(\gamma)(v_0), v_0\rangle| \geq 1 - \epsilon)$, there is a Γ-invariant unit vector $v \in H$ with $\|v - v_0\| < \frac{1}{10}$.

Fix now $\alpha, \beta \in Z^1(a, G)$ and consider the following action of Γ on $X \times G$:

$$\gamma \cdot (x, g) = (\gamma \cdot x, \alpha(\gamma, x) g \beta(\gamma, x)^{-1}).$$

Clearly the σ-finite measure $\mu \times \eta_G$, where η_G is the counting measure on G, is Γ-invariant, and this gives rise to the unitary representation $\pi : \Gamma \to U(L^2(X \times G, \mu \times \eta_G))$, given as usual by $\pi(\gamma)(v)(x,g) = v(\gamma^{-1} \cdot (x,g))$. Consider the vector $v_0 = \chi_{X\times\{1\}}$. Clearly $\|v_0\| = 1$. Also for $\gamma \in \Gamma$,

$$\begin{aligned}\langle \pi(\gamma^{-1})(v_0), v_0\rangle &= \int \chi_{X\times\{1\}}(\gamma \cdot (x,g))\chi_{X\times\{1\}}(x,g)d(\mu \times \eta_G)\\ &= \mu(\{x : \alpha(\gamma,x) = \beta(\gamma,x)\}).\end{aligned}$$

Choose now $\delta > 0$ such that

$$\tilde{d}^1(\alpha,\beta) < \delta \Rightarrow \forall \gamma \in F(\mu(\{x : \alpha(\gamma,x) \neq \beta(\gamma,x)\}) < \epsilon),$$

and thus for $\gamma \in F$,

$$\tilde{d}^1(\alpha,\beta) < \delta \Rightarrow 1 \geq |\langle \pi(\gamma)(v_0), v_0\rangle| \geq 1 - \epsilon.$$

It follows that if $\tilde{d}^1(\alpha,\beta) < \delta$, there is a Γ-invariant unit vector v with $\|v - v_0\| < \frac{1}{10}$. We will use this to find $f : X \to G$ such that $f \cdot \beta = \alpha$, thus $\alpha \sim \beta$.

Now $v \in L^2(X \times G, \mu \times \eta_G)$ and $v(x,g) = v(\gamma \cdot x, \alpha(\gamma,x)g\beta(\gamma,x)^{-1}), \forall \gamma$. Put

$$f_x(g) = |v(x,g)|^2.$$

Then

$$f_{\gamma\cdot x}(\alpha(\gamma,x)g\beta(\gamma,x)^{-1}) = f_x(g)$$

and $g \mapsto \alpha(\gamma,x)g\beta(\gamma,x)^{-1}$ is a permutation of G, so the maps

$$\begin{aligned}V(x) &= \sum_{g\in G} f_x(g),\\ m(x) &= \max_{g\in G} f_x(g)\end{aligned}$$

and

$$k(x) = \mathrm{card}\{g \in G : f_x(g) = m(x)\}$$

are Γ-invariant. By the ergodicity of the Γ-action on X, these functions are constant, a.e. Note that, by Fubini, $V(x) = V = 1$. Also $m(x) = m$ is positive and $\infty > k(x) = k > 0$. If we can show that $k(x) = 1$, then letting

$$f(x) = (\text{the unique } g \text{ such that } f_x(g) = m),$$

we have $f(\gamma \cdot x) = \alpha(\gamma,x)f(x)\beta(\gamma,x)^{-1}$, i.e., $f \cdot \beta = \alpha$.

But if $k \geq 2$, then $|v(x,1)|^2 \leq \frac{1}{2}$, a.e., so

$$\begin{aligned}\|v - v_0\|_2^2 &= \int |v(x,g) - v_0(x,g)|^2 d(\mu \times \eta_G)\\ &\geq \int |v(x,1) - 1|^2 d\mu \geq (\left(1 - \frac{1}{\sqrt{2}}\right)^2 > \frac{1}{100},\end{aligned}$$

a contradiction. □

The preceding and earlier results lead to the following characterization of property (T) groups.

Theorem 30.2. *Let* Γ *be a countable group. Then the following are equivalent:*

(i) Γ *has property* (T),

(ii) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$ *and any countable* G*, the cohomology classes are clopen in* $Z^1(a, G)$*,*

(iii) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$*, there is some Polish* $G \neq \{1\}$ *admitting an invariant metric, such that* $\mathrm{SMOOTH}(a, G) = Z^1(a, G)$*,*

(iv) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$ *and any countable* $G, H^1(a, G)$ *is countable,*

(v) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$*, there is some Polish* $G \neq \{1\}$ *admitting an invariant metric with* $H^1(a, G)$ *countable.*

Proof. (i) $\Rightarrow$ (ii) is 30.1 and (ii) $\Rightarrow$ (iii) is trivial. If (i) fails, then Γ has an ergodic but not E_0-ergodic action a and so, by 27.2, (iii) fails. Finally (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) are clear. □

A special case of this characterization, for abelian G, was proved in Schmidt [Sc3], 3.4. Also note that, as we discussed earlier in Section 27, **(B)**, if Γ has property (T) and G is a torsion-free group with HAP, then $B^1(a, G) = Z^1(a, G)$ for any $a \in \mathrm{ERG}(\Gamma, X, \mu)$.

Remark. There is an analog of 30.1 valid for any countable group Γ, brought to my attention by Ioana; see Ioana [I2]. It is reminiscent of the last remark of Section 14, **(A)**.

Let $a \in A(\Gamma, X, \mu)$, G a countable group with discrete metric d, and put on $Z^1(a, G)$ the complete metric

$$\begin{aligned}\tilde{d}^1_\infty(\alpha, \beta) &= \sup_{\gamma \in \Gamma} \int d(\alpha(\gamma, x), \beta(\gamma, x)) d\mu(x) \\ &= \sup_{\gamma \in \Gamma} \mu(\{x : \alpha(\gamma, x) \neq \beta(\gamma, x)\}).\end{aligned}$$

Then for $a \in \mathrm{ERG}(\Gamma, X, \mu)$, every cohomology class in $Z^1(a, G)$ is clopen in the $\tilde{d}^1_\infty$-topology. Indeed, following the proof of 30.1, we note that for $\gamma \in \Gamma, 1 - \langle \pi(\gamma)(v_0), v_0 \rangle \leq \tilde{d}^1_\infty(\alpha, \beta)$, so $\|\pi(\gamma)(v_0) - v_0\|^2 \leq 2\tilde{d}^1_\infty(\alpha, \beta)$. If v_1 is the unique element of least norm in the closed convex hull of the Γ-invariant set $\{\pi(\gamma)(v_0) : \gamma \in \Gamma\}$, then v_1 is Γ-invariant and $\|v_1 - v_0\| \leq 2\tilde{d}^1_\infty(\alpha, \beta)$. Thus by choosing $\tilde{d}^1_\infty(\alpha, \beta)$ sufficiently small, we can find a Γ-invariant unit vector with $\|v - v_0\| < 1/10$ and repeating the rest of the proof of 30.1, we conclude that $\alpha \sim \beta$.

We can also see that for Γ with property (T), for $a \in \mathrm{ERG}(\Gamma, X, \mu)$ and G countable, the $\tilde{d}^1_\infty$-topology on $Z^1(a, G)$ is the same as the usual topology on $Z^1(a, G)$. This follows from the proof of 30.1.

First notice that if $\|v - v_0\|_2 < \rho < \frac{1}{10}$, then

$$\begin{aligned}\rho^2 > \|v - v_0\|_2^2 &= \int |v(x,g) - \chi_{X\times\{1\}}(x,g)|^2 d(\mu \times \eta_G) \\ &= \sum_{g\neq 1} \int |v(x,g)|^2 d\mu(x) + \int |v(x,1) - 1|^2 d\mu(x) \\ &\geq \sum_{g\neq 1} \tfrac{1}{4}\mu(\{x : |v(x,g)|^2 \geq \tfrac{1}{4}\}) + \tfrac{1}{4}\mu(\{x : |v(x,1)-1|^2 \geq \tfrac{1}{4}\})\end{aligned}$$

so

$$4\rho^2 > \mu(\{x : \exists g \neq 1 (|v(x,g)|^2 \geq \tfrac{1}{2}) \text{ or } |v(x,1) - 1| \geq \tfrac{1}{2}\}),$$

therefore

$$\mu(\{x : \forall g \neq 1(|v(x,g)| < \tfrac{1}{2}) \text{ and } |v(x,1) - 1| < \tfrac{1}{2}\}) > 1 - 4\rho^2,$$

thus $\mu(\{x : f(x) = 1\}) > 1 - 4\rho^2$. We also have $f \cdot \beta = \alpha$, that is, $f(\gamma \cdot x)\beta(\gamma, x)f(x)^{-1} = \alpha(\gamma, x), \forall\gamma$, so $\mu(\{x : \alpha(\gamma, x) \neq \beta(\gamma, x)\}) \leq 8\rho^2, \forall\gamma$, i.e., $\tilde{d}^1_\infty(\alpha, \beta) \leq 8\rho^2$.

Now given $\frac{1}{10} > \rho > 0$ we can choose $\delta > 0$ such that if $\tilde{d}^1(\alpha, \beta) < \delta$, then there is v with $\|v - v_0\|_2 < \rho$, so $\tilde{d}^1_\infty(\alpha, \beta) \leq 8\rho^2$. This shows that the above two topologies coincide.

It also follows from 30.2 and 27.2 that the following are equivalent:

(i) Γ has property (T),

(ii) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$ and countable G, the $\tilde{d}^1_\infty$-topology on $Z^1(a, G)$ coincides with the usual topology.

(iii) For every $a \in \mathrm{ERG}(\Gamma, X, \mu)$ and countable G, $\tilde{d}^1_\infty$ is separable.

(B) One can use 30.1 to derive superrigidity results. First we need the following characterization.

Theorem 30.3 (Popa [Po3]). *Suppose* Γ *is a countable group,* G *a Polish group admitting an invariant metric and* $a \in \mathrm{WMIX}(\Gamma, X, \mu), \alpha \in Z^1(a, G)$. *Consider the product action* $a \times a \in A(\Gamma, X^2, \mu^2)$ *and let*

$$\begin{aligned}\alpha_1(\gamma, (x_1, x_2)) &= \alpha(\gamma, x_1), \\ \alpha_2(\gamma, (x_1, x_2)) &= \alpha(\gamma, x_2),\end{aligned}$$

so that $\alpha_i \in Z^1(a \times a, G), i = 1, 2$. *Then the following are equivalent:*

(i) α *is cohomologous to a group homomorphism* $\varphi : \Gamma \to G$,

(ii) $\alpha_1 \sim \alpha_2$.

Proof (Furman [Fu2]). (i) $\Rightarrow$ (ii): If $\alpha \sim \varphi$, let $f : X \to G$ be such that $\varphi(\gamma) = f(\gamma \cdot x)\alpha(\gamma, x)f(x)^{-1}$. Then if $f_1(x_1, x_2) = f(x_1)$ we have $f_1 \cdot \alpha_1 = \varphi$, so $\alpha_1 \sim \varphi$. Similarly $\alpha_2 \sim \varphi$, thus $\alpha_1 \sim \alpha_2$.

(ii) $\Rightarrow$ (i): Fix $f : X^2 \to G$ with

$$\alpha(\gamma, x_2) = f(\gamma \cdot x_1, \gamma \cdot x_2)\alpha(\gamma, x_1)f(x_1, x_2)^{-1},$$

μ^2-a.e.(x_1, x_2). It follows that

$$\alpha(\gamma, x_2) = f(\gamma \cdot x_1, \gamma \cdot x_2)\alpha(\gamma, x_1)f(x_1, x_2)^{-1},$$
$$\alpha(\gamma, x_2) = f(\gamma \cdot x_3, \gamma \cdot x_2)\alpha(\gamma, x_3)f(x_3, x_2)^{-1},$$

μ^3-a.e.(x_1, x_2, x_3). So

$$\alpha(\gamma, x_1) = f(\gamma \cdot x_1, \gamma \cdot x_2)^{-1} f(\gamma \cdot x_3, \gamma \cdot x_2)\alpha(\gamma, x_3)f(x_3, x_2)^{-1} f(x_1, x_2).$$

Put

$$g(x_1, x_2, x_3) = f(x_1, x_2)^{-1} f(x_3, x_2).$$

Then for the action $a^3 = a \times a \times a \in A(\Gamma, X^3, \mu^3)$, we have

$$g(\gamma \cdot (x_1, x_2, x_3)) = \alpha(\gamma, x_1)g(x_1, x_2, x_3)\alpha(\gamma, x_3)^{-1},$$

μ^3-a.e.(x_1, x_2, x_3).

Claim. *g depends only on x_1, x_3, i.e., there is $\overline{g} : X^2 \to G$ such that*

$$g(x_1, x_2, x_3) = \overline{g}(x_1, x_3)$$

μ^3-a.e.(x_1, x_2, x_3).

Granting this, we also have

$$g(\gamma \cdot x_1, \gamma \cdot x_2, \gamma \cdot x_3) = \overline{g}(\gamma \cdot x_1, \gamma \cdot x_3),$$

μ^3-a.e. (x_1, x_2, x_3), so

$$\overline{g}(\gamma \cdot x_1, \gamma \cdot x_3) = \alpha(\gamma, x_1)\overline{g}(x_1, x_3)\alpha(\gamma, x_3)^{-1},$$

μ^3-a.e. (x_1, x_2, x_3), i.e.,

$$\overline{g}(\gamma \cdot x_1, \gamma \cdot x_3) = \alpha(\gamma, x_1)\overline{g}(x_1, x_3)\alpha(\gamma, x_3)^{-1},$$

μ^2-a.e., (x_1, x_3). Fix x_2^0 such that

$$g(x_1, x_2^0, x_3) = f(x_1, x_2^0)^{-1} f(x_3, x_2^0) = \overline{g}(x_1, x_3),$$

μ^2-a.e. (x_1, x_3), and thus $\overline{g}(\gamma \cdot x_1, \gamma \cdot x_3) = f(\gamma \cdot x_1, x_2^0)^{-1} f(\gamma \cdot x_3, x_2^0)$, μ^2-a.e. (x_1, x_3). Put

$$h(x) = f(x, x_2^0).$$

Then

$$\alpha(\gamma, x_1) = h(\gamma \cdot x_1)^{-1} h(\gamma \cdot x_3)\alpha(\gamma, x_3)h(x_3)^{-1} h(x_1),$$

μ^2-a.e.(x_1, x_3), so

$$h(\gamma \cdot x_1)\alpha(\gamma, x_1)h(x_1)^{-1} = h(\gamma \cdot x_3)\alpha(\gamma, x_3)h(x_3)^{-1},$$

μ^2-a.e.(x_1, x_3). It follows that

$$\varphi(\gamma) = h(\gamma \cdot x)\alpha(\gamma, x)h(x)^{-1}$$

is constant μ-a.e.(x). Clearly then φ is a homomorphism and $\alpha \sim \varphi$.

Proof of the claim. We have for an invariant metric d on G and every γ,

$$d(g(\gamma \cdot (x_1, x_2, x_3)), g(\gamma \cdot (x_1, x_4, x_3))) = d(g(x_1, x_2, x_3), g(x_1, x_4, x_3))$$

μ^4-a.e.(x_1, x_2, x_3, x_4), so

$$(x_1, x_2, x_3, x_4) \mapsto d(g(x_1, x_2, x_3), g(x_1, x_4, x_3))$$

is invariant for the action $a^4 = a \times a \times a \times a \in A(\Gamma, X^4, \mu^4)$. Since a is weak mixing, a^4 is ergodic, so the above function is constant, a.e., say with value d_0. We will show that $d_0 = 0$, i.e.,

$$g(x_1, x_2, x_3) = g(x_1, x_4, x_3)$$

μ^4-a.e.(x_1, x_2, x_3, x_4). Then choose x_4^0 so that $g(x_1, x_2, x_3) = g(x_1, x_4^0, x_3)$, μ^3-a.e.(x_1, x_2, x_3), and put $\overline{g}(x_1, x_3) = g(x_1, x_4^0, x_3)$.

Assume $d_0 > 0$, towards a contradiction, and fix $\{g_n^0\}$ dense in G. Then we have that $\bigcup_n B(g_n^0, d_0/2) = G$. Fix x_1^0, x_3^0 such that

$$d(g(x_1^0, x_2, x_3^0), g(x_1^0, x_4, x_3^0)) = d_0,$$

μ^2-a.e.(x_2, x_4). Let $q(x) = g(x_1^0, x, x_3^0)$. Then if $B_n = q^{-1}(B(g_n^0, d_0/2))$, we have that $\bigcup_n B_n = X$, so for some $n, \mu(B_n) > 0$. Thus $\mu^2(B_n^2) > 0$, so fix $x_2^0, x_4^0 \in B_n$ such that

$$d(g(x_1^0, x_2^0, x_3^0), g(x_1^0, x_4^0, x_3^0)) = d_0,$$

therefore $d(q(x_2^0), q(x_4^0)) = d_0$, contradicting the fact that

$$d(q(x_2^0), g_n^0), d(q(x_4^0), g_n^0)) < \tfrac{d_0}{2}.$$

□

Following Popa [Po3] we call $a \in A(\Gamma, X, \mu)$ *malleable* if for $a^2 = a \times a \in A(\Gamma, X^2, \mu^2)$, there is a continuous path in $C_{a^2} = \{T \in \mathrm{Aut}(X^2, \mu^2) : Ta^2T^{-1} = a^2\}$ from 1 to the flip $(x, y) \mapsto (y, x)$. Note now the following fact.

Proposition 30.4. *Let Γ be a countable group and let $a \in \mathrm{WMIX}(\Gamma, X, \mu)$ be malleable. Let G be a Polish group with an invariant metric and consider $H^1(a^2, G) = Z^1(a^2, G)/_\sim$ with the quotient topology. If $H^1(a^2, G)$ is totally path disconnected (i.e., its path components are trivial), then a is G-superrigid.*

Proof. Fix $\alpha \in Z^1(a, G)$ and consider α_1, α_2 as in 30.3. Let $t \mapsto T_t \in C_{a^2}$ connect 1 with the flip. Now C_{a^2} acts continuously on $Z^1(a^2, G)$ (see Section 24, **(B)**), so $T_t \cdot \alpha_1$ is a continuous path in $Z^1(a^2, G)$ from α_1 to α_2. Projecting to $H^1(a^2, G)$, we conclude that $\alpha_1 \sim \alpha_2$, so α is cohomologous to a homomorphism by 30.3. □

We thus have:

Theorem 30.5 (Popa [Po3]). *Let Γ have property* (T) *and let the action $a \in \mathrm{WMIX}(\Gamma, X, \mu)$ be malleable. Then for any countable G, a is G-superrigid.*

Proof. By 30.1, clearly $H^1(a^2, G)$ is discrete. □

A canonical example of a malleable action of a group Γ is its shift s on $[0, 1]^\Gamma$, where $[0, 1]$ has Lebesgue measure λ. If we identify $T \in \mathrm{Aut}([0, 1]^2, \lambda^2)$ with $T^* \in \mathrm{Aut}(([0, 1]^2)^\Gamma, (\lambda^2)^\Gamma)$, given by $T^*(p) = (\gamma \mapsto T(p(\gamma)))$, then the group $\mathrm{Aut}([0, 1]^2, \lambda^2)$ becomes a closed subgroup of C_{s^2}.

The flip on $[0,1]^\Gamma \times [0,1]^\Gamma = ([0,1]^2)^\Gamma$ is equal to T^*, where T is the flip on $[0,1]^2$. Since $\mathrm{Aut}([0,1]^2, \lambda^2)$ is path connected, there is a continuous path in C_{s^2} from 1 to the flip, so s is malleable.

The previous superrigidity result can be extended as follows.

Theorem 30.6 (Popa [Po3]). *Let Γ be a countable group containing a normal subgroup $\Delta \lhd \Gamma$ with relative property* (T). *Let $a \in \mathrm{Aut}(\Gamma, X, \mu)$ be malleable such that $a|\Delta \in \mathrm{WMIX}(\Delta, X, \mu)$. Then a is G-superrigid for every countable G.*

Proof. First note that for any $b \in \mathrm{ERG}(\Gamma, Y, \nu)$ the proof of 30.1 shows that if $\alpha, \beta \in Z^1(b, G)$ and $\tilde{d}^1(\alpha, \beta) < \delta$, for an appropriate δ, then $\alpha|\Delta \sim \beta|\Delta$. Then the proof of 30.3 shows that if $\alpha \in Z^1(a, G)$ and $\alpha_1|\Delta \sim \alpha_2|\Delta$, then $\alpha|\Delta \sim \varphi$ for some $\varphi \in \mathrm{Hom}(\Delta, G)$. As before the malleability of a implies that there is a path in $Z^1(a^2, G)$ from α_1 to α_2 and thus by applying the preceding observation to $b = a^2 \in \mathrm{ERG}(\Gamma, X^2, \mu^2)$, we see (by cutting this path into small pieces) that $\alpha_1|\Delta \sim \alpha_2|\Delta$, so $\alpha|\Delta \sim \varphi$, for some $\varphi \in \mathrm{Hom}(\Delta, G)$. We will now use that $\Delta \lhd \Gamma$ to find $\psi \in \mathrm{Hom}(\Gamma, G)$ such that $\alpha \sim \psi$.

We have

$$\alpha(\delta, x) = f(\delta \cdot x)\varphi(\delta)f(x)^{-1}, \forall \delta \in \Delta,$$

for some $f \in L(X, \mu, G)$. Put $\beta = f^{-1} \cdot \alpha$, so that $\beta|\Delta = \varphi$. Put

$$\Gamma_0 = \{\gamma \in \Gamma : \beta(\gamma, x) = \beta(\gamma) \text{ is constant a.e.}\}.$$

Clearly $\Delta \subseteq \Gamma_0$. We will show that $\Gamma_0 = \Gamma$, which will complete the proof.

Fix $\gamma \in \Gamma, \delta \in \Delta$. Then $\gamma\delta\gamma^{-1} = \delta' \in \Delta$, so

$$\begin{aligned}\beta(\gamma\delta, x) &= \beta(\gamma, \delta \cdot x)\beta(\delta, x) = \beta(\gamma, \delta \cdot x)\beta(\delta)\\ &= \beta(\delta'\gamma, x) = \beta(\delta', \gamma \cdot x)\beta(\gamma, x) = \beta(\delta')\beta(\gamma, x).\end{aligned}$$

Thus $\beta(\gamma, \delta \cdot x)\beta(\delta) = \beta(\gamma\delta\gamma^{-1})\beta(\gamma, x)$ and so

$$\beta(\gamma, \delta \cdot x) = \beta(\gamma\delta\gamma^{-1})\beta(\gamma, x)\beta(\delta)^{-1}.$$

Therefore for each $\gamma \in \Gamma, A_\gamma = \{(x, y) \in X^2 : \beta(\gamma, x) = \beta(\gamma, y)\}$ is Δ-invariant for the action a^2, so since $a|\Delta$ is weak mixing, and thus $a^2|\Delta$ is ergodic, $\mu^2(A_\gamma) = 0$ or 1. If for all γ it has measure 1, we are done by Fubini. Otherwise for some $\gamma, \beta(\gamma, x) \neq \beta(\gamma, y)$, μ^2-a.e.(x, y), contradicting the fact that for some g_0, $B = \{x : \beta(\gamma, x) = g_0\}$ has positive measure, thus $\mu^2(B) > 0$. □

Here is a sample application of 30.5 to orbit equivalence.

Theorem 30.7 (Popa [Po3]). *Let Γ be a simple countable group with property* (T), *and let s be the shift action of Γ on $X = [0,1]^\Gamma$. Let Δ be a countable group and $a \in \mathrm{FR}(\Delta, Y, \nu)$. If sOEa, then there is an isomorphism $\varphi : \Gamma \to \Delta$ and an isomorphism $q : (X, \mu) \to (Y, \nu)$ such that $q(\gamma \cdot x) = \varphi(\gamma) \cdot q(x)$, i.e., the groups Γ, Δ are isomorphic and identifying Γ, Δ via an isomorphism the actions s, a are isomorphic.*

Proof. Fix an orbit equivalence $p : X \to Y, xE_sy \Leftrightarrow p(x)E_ap(y)$, and let $\alpha : \Gamma \times X \to \Delta$ be the corresponding cocycle: $\alpha(\gamma, x) \cdot p(x) = p(\gamma \cdot x)$. By 30.5 (and the paragraph following it), there is a homomorphism $\varphi : \Gamma \to \Delta$ such that $\alpha \sim \varphi$, so fix Borel $f : X \to \Delta$ such that

$$f(\gamma \cdot x)\alpha(\gamma, x)f(x)^{-1} = \varphi(\gamma).$$

Then if $q(x) = f(x) \cdot p(x)$, we have that $xE_sy \Leftrightarrow q(x)E_aq(y)$ and $q(\gamma \cdot x) = \varphi(\gamma) \cdot q(x)$. Since Γ is simple, φ is an injection and thus q is an injection. Let $A_\delta = f^{-1}(\{\delta\})$. Then $\{A_\delta\}_{\delta\in\Delta}$ is a partition of X and $q(x) = \delta \cdot p(x)$ for $x \in A_\delta$, so $q|A_\delta$ is measure preserving and thus so is q. Thus $q : (X, \mu) \to (Y, \nu)$ is an isomorphism and the proof is complete. □

(C) We conclude with some remarks and questions concerning path connectedness in $H^1(a, G)$. In general it seems interesting to understand under what conditions $H^1(a, G)$ is totally path disconnected.

First, let us note that if Γ is finitely generated, $a \in \mathrm{E_0RG}(\Gamma, X, \mu)$ and G is a countable group in $\mathcal{M}_C$, then if a is G-superrigid, $H^1(a, G)$ is totally path disconnected. To see this notice that there are only countably many homomorphisms from Γ to G, thus $H^1(a, G)$ is countable, while by 29.6 every point in $H^1(a, G)$ is closed. Thus any path in $H^1(a, G)$ must be constant by Sierpinski's Theorem (see Section 27, **(G)**). I do not know if finite generation of Γ is necessary here.

Second, consider a *strongly treeable* countable group Γ, i.e., a group Γ such that for every $a \in \mathrm{FR}(\Gamma, X, \mu)$, E_a is treeable (μ-a.e.) (see Kechris-Miller [KM], Section 30). There are many such groups, e.g., $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$, which admit no non-trivial homomorphism into F_2. Since for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, E_a is treeable, it can be Borel reduced to the free part of the equivalence relation induced by the shift of F_2 on 2^{F_2} (see Jackson-Kechris-Louveau [JKL], 3.5), so the corresponding reduction cocycle is not cohomologous to a homomorphism and thus a is not F_2-superrigid. It follows from 30.4 that if Γ is strongly treeable, with $\mathrm{Hom}(\Gamma, F_2)$ trivial, and s is the shift action of Γ on $[0, 1]^\Gamma$ (which is malleable and isomorphic to $s \times s$), then $H^1(s, F_2)$ is not totally path disconnected.

In fact the preceding arguments also show the following.

Proposition 30.8. *Let* Γ *be a non-amenable, finitely generated group,* G *a countable group in* $\mathcal{M}_C$ *and let* s *be the shift action of* Γ *on* $[0, 1]^\Gamma$*. Then the following are equivalent:*

i) s *is* G*-superrigid,*

ii) $H^1(s, G)$ *is totally path disconnected.*

APPENDIX A

Realifications and complexifications

We will consider below real and complex Hilbert spaces and some connections between them. I would like to thank Scot Adams for a useful conversation on these matters.

If H is a real Hilbert space, we define its *complexification* $H_{\mathbb{C}}$ as follows: The elements of $H_{\mathbb{C}}$ are the formal sums $x+i\cdot y$ (so, as a set, $H_{\mathbb{C}} = H^2$) with vector operations defined in the obvious way: $(x_1 + i \cdot y_1) + (x_2 + i \cdot y_2) = (x_1 + x_2) + i \cdot (y_1 + y_2), (a + bi)(x + i \cdot y) = (ax - by) + i \cdot (bx + ay)$, for $a, b \in \mathbb{R}$. The inner product is again defined by

$$\langle x_1 + i \cdot y_1, x_2 + i \cdot y_2 \rangle_{H_{\mathbb{C}}} = \langle x_1, x_2 \rangle_H + \langle y_1, y_2 \rangle_H + i\langle y_1, x_2 \rangle_H - i\langle x_1, y_2 \rangle_H.$$

We will write $H_{\mathbb{C}} = H + i \cdot H$, and identify $x \in H$ with $x + i \cdot 0$. The norm of $x + i \cdot y$ in $H_{\mathbb{C}}$ is $\|x + i \cdot y\|^2_{H_{\mathbb{C}}} = \langle x + i \cdot y, x + i \cdot y \rangle_{H_{\mathbb{C}}} = \langle x, x \rangle_H + \langle y, y \rangle_H + i\langle y, x \rangle_H - i\langle x, y \rangle_H = \|x\|^2_H + \|y\|^2_H$. It follows that $H_{\mathbb{C}}$ is a complex Hilbert space. If $\{e_\alpha\}$ is an orthonormal basis in H, then $\{e_\alpha\}$ is also an orthonormal basis for $H_{\mathbb{C}}$.

If H is a complex Hilbert space, we define its *realification* $H_{\mathbb{R}}$ by restriction of scalars. The elements of $H_{\mathbb{R}}$ are exactly those of H and the vector operations are defined by restriction to $\mathbb{R}$. The inner product is given by

$$\langle x, y \rangle_{H_{\mathbb{R}}} = \text{Re } \langle x, y \rangle_H.$$

Thus $\|x\|^2_{H_{\mathbb{R}}} = \text{Re } \langle x, x \rangle_H = \langle x, x \rangle_H = \|x\|^2_H$, so that the norms are the same and $H_{\mathbb{R}}$ is a real Hilbert space. If $\{e_\alpha\}$ is an orthonormal basis for H, then $\{e_\alpha, ie_\alpha\}$ is an orthonormal basis for $H_{\mathbb{R}}$.

If H is a real Hilbert space, the identity map is a Hilbert space embedding of H into $(H_{\mathbb{C}})_{\mathbb{R}}$ (i.e., a real Hilbert space isomorphism of H with a closed subspace on $(H_{\mathbb{C}})_{\mathbb{R}}$).

If H is a complex Hilbert space, then we can define a canonical Hilbert space embedding of H into $(H_{\mathbb{R}})_{\mathbb{C}}$ as follows:

$$\varphi(x) = \frac{1}{\sqrt{2}}(x + i \cdot (-ix))$$

If H is a real Hilbert space we denote by $O(H)$ the *orthogonal group* of H, i.e., the group of Hilbert space isomorphisms of H. We equip $O(H)$ with the *strong operator topology*, i.e., the one induced by the maps $T \mapsto T(x)$ from $O(H)$ to H for all $x \in H$. This is the same as the *weak operator topology*, i.e., the one induced by the maps $T \mapsto \langle T(x), y \rangle$ from $O(H)$ to $\mathbb{R}$

for all $x, y \in H$. If H is separable, with this topology, $O(H)$ is a Polish group.

If H is a complex Hilbert space, we similarly define the *unitary group* of $H, U(H)$.

If H is a real Hilbert space and $T \in O(H)$, let $T_{\mathbb{C}} = T + i \cdot T$ be the element of $U(H_{\mathbb{C}})$ given by

$$(T + i \cdot T)(x + i \cdot y) = T(x) + i \cdot T(y).$$

Clearly $T_{\mathbb{C}}$ is the unique $T_1 \in U(H_{\mathbb{C}})$ with $T_1|H = T$ and $\{T_{\mathbb{C}} : T \in O(H)\} = \{T_1 \in U(H) : T_1(H) = H\}$, so via $T \mapsto T_{\mathbb{C}}$ (which is easily a topological group isomorphism) we can identify $O(H)$ with the closed subgroup of $U(H_{\mathbb{C}})$ consisting of elements that leave H invariant, so $O(H) \subseteq U(H_{\mathbb{C}})$.

If H is a complex Hilbert space and $T \in U(H)$, clearly $T \in O(H_{\mathbb{R}})$ and the identity map is a topological group isomorphism of $U(H)$ onto a closed subgroup of $O(H_{\mathbb{R}})$, thus we have again $U(H) \subseteq O(H_{\mathbb{R}})$.

If H is a real Hilbert space and we view H, via the identity map, as a closed subspace of $(H_{\mathbb{C}})_{\mathbb{R}}$, then, via $T \mapsto T + i \cdot T, O(H)$ is identified with a closed subgroup of $O((H_{\mathbb{C}})_{\mathbb{R}})$, so $O(H) \subseteq O((H_{\mathbb{C}})_{\mathbb{R}})$.

If H is a complex Hilbert space, $T \in U(H)$ and we identify H with a closed subspace $\tilde{H}$ of $(H_{\mathbb{R}})_{\mathbb{C}}$ via $x \mapsto \frac{1}{\sqrt{2}}(x + i \cdot (-ix))$, then every $T \in U(H)$ gives rise to $\tilde{T} \in U(\tilde{H})$ defined by $\tilde{T}(\frac{1}{\sqrt{2}}(x + i \cdot (-ix)) = \frac{1}{\sqrt{2}}(T(x) + i \cdot (-iT(x)))$. But also T gives rise to $T + i \cdot T \in U((H_{\mathbb{R}})_{\mathbb{C}})$. We note that $T + i \cdot T$ extends $\tilde{T}$ from $\tilde{H}$ to $(H_{\mathbb{R}})_{\mathbb{C}}$. Thus if we identify $T \in U(H)$ with $T + i \cdot T \in U((H_{\mathbb{R}})_{\mathbb{C}})$, then $T + i \cdot T$ leaves $\tilde{H}$ invariant and $(T + i \cdot T)|\tilde{H} = \tilde{T}$.

Thus we can view $U(H)$ as either $U(\tilde{H})$ via the identification $T \mapsto \tilde{T}$ or as a closed subgroup of $U((H_{\mathbb{R}})_{\mathbb{C}})$ via the identification $T \mapsto T + i \cdot T$. The above shows that these identifications cohere, in the sense that $\tilde{H}$ is invariant under each $T + i \cdot T$ and its restriction to $\tilde{H}$ gives $\tilde{T}$.

APPENDIX B

Tensor products of Hilbert spaces

Below we consider either real or complex vector and Hilbert spaces.

If $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are vector spaces, their (algebraic) tensor product is the (unique up to isomorphism) vector space $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ equipped with a multilinear map $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n, (v_1, \ldots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n$ such that for any vector space $\mathcal{V}$ and multilinear map $\pi : \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{V}$ there is unique linear map $\rho : \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n \to \mathcal{V}$ with $\pi(v_1, \ldots, v_n) = \rho(v_1 \otimes \cdots \otimes v_n)$. Every element of $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ is a linear combination of vectors of the form $v_1 \otimes \cdots \otimes v_n$. If $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are bases for $\mathcal{V}_1, \ldots, \mathcal{V}_n$, resp., then $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n = \{v_1 \otimes \cdots \otimes v_n : v_i \in \mathcal{B}_i\}$ is a basis for $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$.

Now assume that $H_1, \ldots, H_n$ are Hilbert spaces. Then there is a unique inner product on the algebraic tensor product $H_1 \otimes \cdots \otimes H_n$ such that

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle_{H_1 \otimes \cdots \otimes H_n} = \prod_{i=1}^{n} \langle v_i, w_i \rangle_{H_i}.$$

Then we define the Hilbert space tensor product, also denoted by $H_1 \otimes \cdots \otimes H_n$ as the completion of this inner product space. (We have used $H_1 \otimes \cdots \otimes H_n$ to denote both the algebraic and the Hilbert space tensor product. Since it is the latter we are interested in, we will use this notation only for the latter in the sequel.) Note that the linear combinations of the vectors $v_1 \otimes \cdots \otimes v_n, v_i \in H_i$ are dense in $H_1 \otimes \cdots \otimes H_n$. Also $\|v_1 \otimes \cdots \otimes v_n\|_{H_1 \otimes \cdots \otimes H_n} = \|v_1\|_{H_i} \cdots \|v_n\|_{H_n}$. Finally, if $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are orthonormal bases for $H_1, \ldots, H_n$, resp., then $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$ is an orthonormal basis for $H_1 \otimes \cdots \otimes H_n$.

If $H_1 = \cdots = H_n = H$, we let $H^{\otimes n} = H_1 \otimes \cdots \otimes H_n$. By convention $H^{\otimes 0}$ = the scalar field (viewed as a 1-dimensional Hilbert space) and $H^{\otimes 1} = H$.

We next define the *symmetric tensor power* of H. Note that for each $\sigma \in S_n$ (= the permutation group of $\{1, \ldots, n\}$) there is a unique operator on $H^{\otimes n}$ also denoted by σ such that $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. Let $H^{\odot n}$ be the (closed) subspace of all vectors in $H^{\otimes n}$ invariant under all $\sigma \in S_n$. Also $H^{\odot n}$ can be viewed as the range (or equivalently the set of fixed points) of the operator $\tau = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$. Let $v_1 \odot \cdots \odot v_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} = \sqrt{n!}\tau(v_1 \otimes \cdots \otimes v_n)$. Then the linear combinations of the vectors $v_1 \odot \cdots \odot v_n, v_i \in H$, are dense in $H^{\odot n}$. Moreover $\langle v_1 \odot \cdots \odot v_n, w_1 \odot \cdots \odot w_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle v_i, w_{\sigma(i)} \rangle$, so that if $\mathcal{B}$ is an orthonormal basis for H, then

$\mathcal{B} \odot \cdots \odot \mathcal{B}$ is an orthonormal basis for $H^{\odot n}$. We finally have $H^{\odot 0} =$ scalars, $H^{\odot 1} = H$.

The direct sum

$$H^{\odot\infty} = H^{\odot 0} \oplus H^{\odot 1} \oplus \cdots \oplus H^{\odot n} \oplus \dots$$

is called the *symmetric Fock space* or *boson Fock space* of H.

Any bounded linear map $L : H_1 \to H_2$ between Hilbert spaces extends to a unique bounded linear map $L^{\otimes n} : H_1^{\otimes n} \to H_2^{\otimes n}$, so that

$$L^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = L(v_1) \otimes \cdots \otimes L(v_n)$$

and similarly $L^{\odot n} : H_1^{\odot n} \to H_2^{\odot n}$ so that

$$L^{\odot n}(v_1 \odot \cdots \odot v_n) = L(v_1) \odot \cdots \odot L(v_n).$$

We have $\|L^{\otimes n}\| = \|L^{\odot n}\| = \|L\|^n$. Finally, if $\|L\| \leq 1$, let $L^{\odot\infty} = \bigoplus_{n=0}^{\infty} L^{\odot n}$, which is a bounded linear map from $H_1^{\odot\infty}$ to $H_2^{\odot\infty}$.

In particular, If T is an orthogonal (resp., unitary) operator on a real (resp. complex) Hilbert space H, then T induces a unique orthogonal (resp., unitary) operator $T^{\otimes n}$ on $H^{\otimes n}$ so that

$$T^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = T(v_1) \otimes \cdots \otimes T(v_n),$$

and similarly $T^{\odot n}$ on $H^{\odot n}$ so that

$$T^{\odot n}(v_1 \odot \cdots \odot v_n) = T(v_1) \odot \cdots \odot T(v_n).$$

Finally let $T^{\odot\infty}$ be the operator $\bigoplus_{n=0}^{\infty} T^{\odot n}$ on $H^{\odot\infty}$. The map $T \mapsto T^{\odot\infty}$ is an isomorphism of $O(H)$ (resp., $U(H)$) with a closed subgroup of $O(H^{\odot\infty})$ (resp., $U(H^{\odot\infty})$). Clearly $T^{\odot\infty}|H = T$.

Comments. We have followed here the exposition of tensor products in Janson [Ja], Appendix E.

APPENDIX C

Gaussian probability spaces

Suppose F is a finite (non-$\emptyset$) set and $\varphi : F^2 \to \mathbb{R}$ a real *positive-definite function* on F, i.e., a symmetric function ($\varphi(i,j) = \varphi(j,i)$), satisfying $\sum_{i,j \in F} a_i a_j \varphi(i,j) \geq 0$ for all reals $a_i, i \in F$. Then a standard result in probability theory asserts that there is a unique probability Borel measure μ_φ on $\mathbb{R}^F$ such that the sequence of projection functions $p_i(x) = x_i$, where $x = (x_i)_{i\in F}$, from $\mathbb{R}^F$ to $\mathbb{R}$ is a $\mathbb{R}^d$-valued (where $d = \text{card}(F)$) Gaussian random variable with mean 0 and covariance matrix φ. This means that the following two conditions are satisfied:

(i) Every linear combination $f = \sum_{i\in F} \alpha_i p_i : \mathbb{R}^F \to \mathbb{R}$ is a centered Gaussian random variable, i.e., has centered Gaussian distribution $N(0, \sigma^2)$, for some σ. Explicitly this means that $f_*\mu_\varphi$ is the centered Gaussian measure with density $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ $(-\infty < x < \infty), \sigma \geq 0$. When $\sigma = 0$, we interpret $N(0,0)$ as the Dirac (point) measure at 0.

(ii) $\text{Cor}(p_i p_j) = \mathbb{E}(p_i p_j) = \int x_i x_j d\mu_\varphi = \varphi(i,j)$, for $(i,j) \in F^2$.

Note that the projection of μ_φ to the ith coordinate is the centered Gaussian measure $N(0, \varphi(i,i))$. The characteristic function of this measure, which is defined by $\tilde{\mu}_\varphi(u) = \int e^{i(\sum_{k\in F} u_k x_k)} d\mu_\varphi(x), \tilde{\mu}_\varphi : \mathbb{R}^F \to \mathbb{C}$, is given by the following formula

$$\tilde{\mu}_\varphi(u) = e^{-\frac{1}{2}\sum_{i,j\in F} u_i u_j \varphi(i,j)}.$$

Recall that $\tilde{\mu}_\varphi$ uniquely determines μ_φ. We call μ_φ the *Gaussian centered measure* associated to φ. For example, if $\varphi(i,j) = \delta_{ij}$ (= the Kronecker delta), then μ_φ is the product measure on $\mathbb{R}^F$, where in each coordinate we have the standard Gaussian measure with distribution $N(0,1)$ (i.e., density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$).

Now consider a countable set I. Using the preceding and the Kolmogorov Consistency Theorem, we see that for each real positive-definite $\varphi : I^2 \to \mathbb{R}$ (i.e., $\varphi(i,j) = \varphi(j,i), \sum_{i,j\in F} a_i a_j \varphi(i,j) \geq 0$, for all finite $F \subseteq I$ and reals $a_i, i \in F$), there is a unique probability Borel measure μ_φ on $\mathbb{R}^I$ such that for each finite (non-$\emptyset$) $F \subseteq I$, if $\pi_F : \mathbb{R}^I \to \mathbb{R}^F$ is the canonical projection $\pi_F(x) = x|F$ $(x \in \mathbb{R}^I)$, then $(\pi_F)_*\mu_\varphi = \mu_{\varphi|F}$. Thus μ_φ is uniquely determined by saying that each linear combination $f = \sum_{i\in F} \alpha_i p_i : \mathbb{R}^I \to \mathbb{R}$, where $p_i(x) = x_i$, is a centered Gaussian random variable and $\mathrm{Cor}(p_i p_j) = \mathbb{E}(x_i x_j) = \int x_i x_j d\mu_\varphi = \varphi(i,j)$. We call this the *Gaussian centered measure* associated to φ. Again if $\varphi(i,j) = \delta_{i,j}$, μ_φ is the product measure μ^I, where $\mu = N(0,1)$.

A special case arises as follows: Let Γ be a countable group and $\varphi : \Gamma \to \mathbb{R}$ a real *positive-definite function* on Γ, i.e., one for which the function $\varphi' : \Gamma^2 \to \mathbb{R}$ given by

$$\varphi'(\gamma, \delta) = \varphi(\gamma^{-1}\delta)$$

is positive definite. This means that $\varphi(\gamma) = \varphi(\gamma^{-1})$ and for any finite $F \subseteq \Gamma$ and any reals $a_\gamma, \gamma \in F$, we have $\sum_{\gamma,\delta \in F} a_\gamma a_\delta \varphi(\gamma^{-1}\delta) \geq 0$. Consider then the measure $\mu_{\varphi'}$ on $\mathbb{R}^\Gamma$ and the shift action of Γ on $\mathbb{R}^\Gamma$:

$$\gamma \cdot f(\delta) = f(\gamma^{-1}\delta).$$

Using the characterization of $\mu_{\varphi'}$ in terms of the properties of the projection function $p_\gamma(x) = x_\gamma$, it is easy to check that this action preserves $\mu_{\varphi'}$. For simplicity we will write μ_φ instead of $\mu_{\varphi'}$ when $\varphi : \Gamma \to \mathbb{R}$ is positive-definite. We will call the shift action of Γ on $(\mathbb{R}^\Gamma, \mu_\varphi)$ the *Gaussian shift* (associated with φ).

Comments. Some references here are Jacod-Protter [JP], §16, Glasner [Gl2], pp. 89–91, Khoshnevisan [Kh], §5.

APPENDIX D

Wiener chaos decomposition

(A) Let I be a countable set, φ a real positive-definite function $\varphi : I^2 \to \mathbb{R}$ and consider the Gaussian space $(\mathbb{R}^I, \mu_\varphi)$. There is a canonical decomposition of $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$ (the real Hilbert space of real-valued square integrable functions on $(\mathbb{R}^I, \mu_\varphi)$) called the *Wiener chaos decomposition*.

Let H = the closed linear span of the projection functions $p_i(x) = x_i, i \in I$. Then it is not hard to see that every $f \in H$ has centered Gaussian distribution (i.e., $f_*\mu_\varphi$ has distribution $N(0, \sigma^2)$, for some σ^2). (Such closed linear subspaces of L^2 are called (real) *Gaussian Hilbert spaces*.) Next for each $n \geq 0$, let $\bar{P}_n(H)$ be the closure in $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$ of the linear subspace $P_n(H)$ consisting of all $p(f_1, \dots, f_m), p$ a real polynomial of degree $\leq n$ (in m variables for some m), and $f_1, \dots, f_m \in H$. Then $\bar{P}_0(H) = \mathbb{R}$ (the constant functions) $\subseteq \bar{P}_1(H) \subseteq \bar{P}_2(H) \subseteq \bar{P}_3(H) \subseteq \dots$ and we let (following standard notation)

$$H^{:0:} = \mathbb{R},$$

$$H^{:n:} = \bar{P}_n(H) \ominus \bar{P}_{n-1}(H) = \bar{P}_n(H) \cap \bar{P}_{n-1}(H)^\perp, n \geq 1.$$

Then (noticing that $H \perp \mathbb{R}$ as the projection functions have mean 0) we have that $H^{:1:} = H, \{H^{:n:}\}_{n \geq 0}$ are pairwise orthogonal and (using that the projection functions separate points, so they generate the Borel algebra of $\mathbb{R}^I$) that $\bigcup_n P_n(H) = P(H)$ is dense in $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$. Therefore

$$L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}) = \bigoplus_{n \geq 0} H^{:n:}.$$

This is called the *Wiener chaos decomposition* of $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$. $H = H^{:1:}$ is called the *first chaos*.

(B) This decomposition has an alternate description using symmetric tensor products. There is a canonical Hilbert space isomorphism of $H^{\odot n}$ with $H^{:n:}$. This isomorphism sends $f_1 \odot \cdots \odot f_n$ (where $f_i \in H$) to the so-called *Wick product* of $f_1, \dots, f_n$ denoted by $: f_1 \dots f_n :$ which is by definition the projection of the product $f_1 \dots f_n$ into the space $H^{:n:}$. Thus under this identification of $H^{\odot n}$ with $H^{:n:}$, we can also write

$$H^{\odot \infty} = \bigoplus_{n \geq 0} H^{\odot n} = \bigoplus_{n \geq 0} H^{:n:} = L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}).$$

(C) Suppose now φ, ψ are real positive-definite functions and we denote by H_φ, H_ψ the corresponding first Wiener chaos for φ, ψ, resp. Let

$T : H_\varphi \to H_\psi$ be a Hilbert space isomorphism. Then, since T extends canonically to the symmetric Fock space of H_φ (see Appendix B), there is a canonical isomorphism $T^{\odot\infty} : H_\varphi^{\odot\infty} = L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}) \to H_\psi^{\odot\infty} = L^2(\mathbb{R}^I, \mu_\psi, \mathbb{R})$ extending T. We will actually see that there is a (unique) isomorphism $S : (\mathbb{R}^I, \mu_\varphi) \to (\mathbb{R}^I, \mu_\psi)$ which gives T in the sense that the corresponding isomorphism $O_S : L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}) \to L^2(\mathbb{R}^I, \mu_\psi, \mathbb{R})$ induced by S, i.e., $O_S(f) = f \circ S^{-1}$, is equal to $T^{\odot\infty}$, i.e., $T^{\odot\infty}(f) = f \circ S^{-1}$, $f \in L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$. To prove this, consider the projection functions $p_i(x) = x_i$ on $\mathbb{R}^I$ viewed as elements of H_φ and let $T(p_i) = q_i \in H_\psi$. Then let $\theta : \mathbb{R}^I \to \mathbb{R}^I$ be given by $\theta(x)_i = q_i(x)$. We claim that θ is 1-1 on a μ_ψ-measure 1 set and $\theta_*\mu_\psi = \mu_\varphi$. Thus $\theta : (\mathbb{R}^I, \mu_\psi) \to (\mathbb{R}^I, \mu_\varphi)$ is an isomorphism. Let $S = \theta^{-1}$. Then $O_S(p_i)(x) = p_i(\theta(x)) = \theta(x)_i = q_i(x)$, i.e., $O_S(p_i) = q_i$, so $O_S|H_\varphi = T$ and, since O_S preserves multiplication, it follows that $O_S = T^{\odot\infty}$.

To see that θ is 1-1 on a μ_ψ-measure 1 set, note that the closed linear span of $\{q_i\}$ contains each p_i, thus the σ-algebra generated by $\{q_i\}$ is the Borel σ-algebra modulo μ_ψ-null sets, so there is a Borel μ_ψ-conull set A such that $\{q_i|A\}$ generates the Borel σ-algebra of A and thus $\{q_i|A\}$ separates points and therefore θ is 1-1 on A.

We will finally verify that $\theta_*\mu_\psi = \mu_\varphi$. For this it is enough to check that $(\mathbb{R}^I, \theta_*\mu_\psi)$ is the Gaussian centered measure space associated to φ. Translated back to $(\mathbb{R}^I, \mu_\psi)$ this means that every linear combination $f = \sum_{i=1}^n \alpha_i q_i$ has centered Gaussian distribution, which is clear as $f \in H_\psi$, and that $\mathbb{E}_{\mu_\psi}(q_i q_j) = \varphi(i,j)$, which is clear as $\mathbb{E}_{\mu_\psi}(q_i q_j) = \langle q_i, q_j \rangle_{H_\psi} = \langle T(p_i), T(p_j) \rangle_{H_\psi} = \langle p_i, p_j \rangle_{H_\varphi} = \mathbb{E}_{\mu_\varphi}(p_i p_j) = \varphi(i,j)$.

In particular, if H is the first chaos of $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$ and $T \in O(H) =$ the orthogonal group of H, then $T^{\odot\infty} \in O(H^{\odot\infty}) = O(L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}))$ extends T canonically to $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$ and there is a unique $S \in \mathrm{Aut}(\mathbb{R}^I, \mu_\varphi)$ so that if $O_S \in O(L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}))$ is the associated orthogonal operator given by $O_S(f) = f \circ S^{-1}$, then $O_S|H = T$ and $O_S = T^{\odot\infty}$. Thus identifying T and S we can view $O(H)$ as the closed subgroup of $\mathrm{Aut}(\mathbb{R}^I, \mu_\varphi)$ (with the weak topology) consisting of all $S \in \mathrm{Aut}(\mathbb{R}^I, \mu_\varphi)$ for which $O_S(H) = H$.

Note that the complex Hilbert space $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{C})$ is in the obvious sense the complexification of $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R})$

$$L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{C}) = L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}) + i \cdot L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{R}).$$

Also we have the decomposition (under again some obvious identifications)

$$\begin{aligned} H_{\mathbb{C}}^{\odot\infty} = H^{\odot\infty} + i \cdot H^{\odot\infty} = \textstyle\bigoplus_{n\geq 0} H_{\mathbb{C}}^{:n:} &= \textstyle\bigoplus_{n\geq 0} (H_{\mathbb{R}}^{:n:} + i \cdot H_{\mathbb{R}}^{:n:}) \\ &= L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{C}). \end{aligned}$$

Moreover if we define $P_{n,\mathbb{C}}(H)$ by using complex polynomials of degree $\leq n$, then we have

$$H_{\mathbb{C}}^{:n:} = \overline{P_{n,\mathbb{C}}}(H) \ominus \overline{P_{n-1,\mathbb{C}}}(H),$$

where the closure is now in $L^2(\mathbb{R}^I, \mu_\varphi, \mathbb{C})$. (Note also that $H_{\mathbb{C}}^{:0:} = \mathbb{C}$.)

(D) Now consider the special case where Γ is a countable group and $\varphi : \Gamma \to \mathbb{R}$ is a real positive-definite function, and the corresponding shift action of Γ on $(\mathbb{R}^\Gamma, \mu_\varphi)$. Let $p_1(x) = x_1$ be the projection to the identity element of Γ ($p_1 : \mathbb{R}^\Gamma \to \mathbb{R}$). Considering the orthogonal Koopman representation of Γ on $L^2(\mathbb{R}^\Gamma, \mu_\varphi, \mathbb{R})$ associated to the shift (i.e., $\gamma \cdot f(x) = f(\gamma^{-1} \cdot x)$), we have $\gamma \cdot p_1(x) = p_1(\gamma^{-1} \cdot x) = (\gamma^{-1} \cdot x)_1 = x_\gamma = p_\gamma(x)$, i.e., $\gamma \cdot p_1 = p_\gamma$. Thus

$$H = \langle p_\gamma \rangle_{\gamma \in \Gamma} = \langle \gamma \cdot p_1 \rangle_{\gamma \in \Gamma},$$

where $\langle X \rangle$ denotes the closed linear span of X. Moreover,

$$\langle \gamma \cdot p_1, p_1 \rangle = \langle p_\gamma, p_1 \rangle = \mathbb{E}(p_\gamma p_1) = \varphi(\gamma^{-1}) = \varphi(\gamma)$$

(as every real positive-definite function is symmetric).

Recall that given a real positive-definite function φ on Γ there is a unique (up to the obvious notion of isomorphism) triple $(H_\varphi, \pi_\varphi, e_\varphi)$ consisting of a real Hilbert space H_φ, an orthogonal representation π_φ of Γ on H_φ, and a *cyclic* vector $e_\varphi \in H_\varphi$ (i.e., $\langle \pi_\varphi(\gamma)(e_\varphi) \rangle_{\gamma \in \Gamma} = H_\varphi$) such that $\varphi(\gamma) = \langle \pi_\varphi(\gamma)(e_\varphi), e_\varphi \rangle$. This is the well-known GNS construction. It follows that the first Wiener chaos H with the restriction of the orthogonal representation induced by the shift action and p_1 is the GNS representation associated to φ.

(E) Suppose now that φ_1 is the characteristic function of $\{1\} \subseteq \Gamma$. Clearly this is positive-definite. The corresponding μ_{φ_1} is simply the product μ^Γ, where μ = normalized Gaussian measure on $\mathbb{R} = N(0,1)$, so we are looking at the shift action of Γ on $(\mathbb{R}^\Gamma, \mu^\Gamma)$. Then $\{p_\gamma\}_{\gamma \in \Gamma}$ clearly is an orthonormal basis for H on which Γ acts transitively: $\gamma \cdot p_\delta = p_{\gamma\delta}$. So the orthogonal representation of Γ on H is simply (isomorphic to) the (left-) regular representation of Γ on $\ell^2(\Gamma, \mathbb{R})$.

For each finite multiset $F \subseteq \Gamma$ of size $n \geq 2$, i.e., an object of the form $F = \{n_1\gamma_1, \dots, n_k\gamma_k\}$, where $n_i \geq 1, \sum_{i=1}^k n_k = n, \gamma_1, \dots, \gamma_k \in \Gamma$ distinct, let $p^{\odot F} = p_{\gamma_1} \odot \cdots \odot p_{\gamma_1} \odot \cdots \odot p_{\gamma_k} \odot \cdots \odot p_{\gamma_k}$, where γ_i is repeated n_i times. Then for each $n \geq 2$, $\{p^{\odot F} :$ size of $F = n\}$ is an orthonormal basis for $H^{\odot n}$. Denote by $(\Gamma)^n$ the set of all multisets of size n. Then Γ acts on $(\Gamma)^n$ by $\gamma \cdot \{n_1\gamma_1, \dots, n_k\gamma_k\} = \{n_1\gamma\gamma_1, \dots, n_k\gamma\gamma_k\}$. Let $\mathcal{O}_1^{(n)}, \mathcal{O}_2^{(n)}, \dots$ be the orbits of this action. Choose $F_i^{(n)} \in \mathcal{O}_i^{(n)}$ and let $\Gamma_i^{(n)} = \{\gamma : \gamma \cdot F_i^{(n)} = F_i^{(n)}\}$, which is a finite subgroup of Γ. Then clearly $\langle p^{\odot F} : F \in \mathcal{O}_i^{(n)} \rangle = H_i^{(n)}$ is a closed subspace of $H^{\odot n}$ invariant under the orthogonal Koopman representation and the representation of Γ on $H_i^{(n)}$ is isomorphic to the quasi-regular representation of Γ on $\ell^2(\Gamma/\Gamma_i^{(n)}, \mathbb{R})$ (given by $\gamma \cdot f(\delta\Gamma_i^{(n)}) = f(\gamma^{-1}\delta\Gamma_i^{(n)})$). Thus the orthogonal representation of Γ on $L^2(\mathbb{R}^\Gamma, \mu^\Gamma, \mathbb{R})$ associated to the shift action is isomorphic to $1_{\Gamma,\mathbb{R}} \oplus \lambda_{\Gamma,\mathbb{R}} \oplus \bigoplus_{n,i} \lambda_{\Gamma/\Gamma_i^{(n)},\mathbb{R}}$ where $1_{\Gamma,\mathbb{R}}$ is the trivial 1-dimensional orthogonal representation, $\lambda_{\Gamma,\mathbb{R}}$ is the regular orthogonal representation on $\ell^2(\Gamma, \mathbb{R})$ and $\lambda_{\Gamma/\Delta,\mathbb{R}}$ is the quasi-regular orthogonal representation on $\ell^2(\Gamma/\Delta, \mathbb{R})$. Complexifying we have that if κ is the unitary Koopman representation (on $L^2(\mathbb{R}^\Gamma, \mu^\Gamma, \mathbb{C})$) of the shift action of Γ, then

$$\kappa \cong 1_\Gamma \oplus \lambda_\Gamma \oplus \bigoplus_{n,i} \lambda_{\Gamma/\Gamma_i^{(n)}},$$

where we are now referring to unitary representations (on the spaces $\mathbb{C}$, $\ell^2(\Gamma, \mathbb{C})$, $\ell^2(\Gamma/\Gamma_i^{(n)}, \mathbb{C})$, resp.). If Γ is torsion-free, so that $\Gamma_i^{(n)} = \{1\}$, then

$$\kappa \cong 1_\Gamma \oplus \infty \cdot \lambda_\Gamma.$$

(where $\infty \cdot \rho$ means the sum of $\aleph_0$ many copies of ρ). In general, if $\kappa_0 = \kappa | L_0^2(\mathbb{R}^\Gamma, \mu^\Gamma, \mathbb{C}))$, so that $\kappa = 1_\Gamma \oplus \kappa_0$, it can be shown that $\kappa_0 \leq \infty \cdot \lambda_\Gamma$ (see Bekka-de la Harpe-Valette [BdlHV], E.4.5).

Comments. For the Wiener chaos decomposition and Gaussian Hilbert spaces, see Janson [Ja].

APPENDIX E

Extending representations to actions

Let H be a separable real Hilbert space of dimension $1 \leq N \leq \infty$. Consider the product space $(\mathbb{R}^N, \mu^N)$, where μ is the standard Gaussian measure with distribution $N(0,1)$. Then if $\{p_i\}$ are the projection functions, $p_i(x) = x_i$, the first Wiener chaos $H^{:1:} = \langle p_i \rangle$ has dimension N, as $\{p_i\}$ is an orthonormal basis for it. Fix an isomorphism $\theta : H \cong H^{:1:}$. Consider now $T \in O(H)$ and $\theta T \theta^{-1} \in O(H^{:1:})$. Then there is unique $S \in \mathrm{Aut}(\mathbb{R}^N, \mu^N)$ such that $O_S | H^{:1:} = \theta T \theta^{-1}$. We can identify T with S and simply view $O(H)$ as a closed subgroup of $\mathrm{Aut}(\mathbb{R}^N, \mu^N)$. (This identification of course depends on θ.) It follows that an orthogonal representation π of a countable group Γ on H gives rise to an action $a_\pi \in A(\Gamma, \mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$. Identifying H and $H^{:1:}$ via θ, it is clear that π is the restriction of the orthogonal Koopman representation associated with a_π to the space $H^{:1:}$.

Fix now a separable complex Hilbert space H. Let $H_\mathbb{R}$ be its realification and let N be the dimension of $H_\mathbb{R}$ (as a real Hilbert space). Consider again the product space $(\mathbb{R}^N, \mu^N)$. Fix an isomorphism $\theta : H_\mathbb{R} \cong H^{:1:}$. Then θ extends to an isomorphism, $\theta_\mathbb{C} : (H_\mathbb{R})_\mathbb{C} \cong H^{:1:}_\mathbb{C}$ of the complexifications of $H_\mathbb{R}, H^{:1:}$. Consider now $T \in U(H)$. Then $T \in O(H_\mathbb{R})$ and $\theta T \theta^{-1} \in O(H^{:1:})$. Thus there is unique $S \in \mathrm{Aut}(\mathbb{R}^N, \mu^N)$ such that $O_S | H^{:1:} = \theta T \theta^{-1}$. We can identify again T with S and simply view $U(H)$ as a closed subgroup of $\mathrm{Aut}(\mathbb{R}^N, \mu^N)$. It follows that a unitary representation $\pi \in \mathrm{Rep}(\Gamma, H)$ of a countable group Γ on H gives rise to an action $a_\pi \in A(\Gamma, \mathbb{R}^N, \mu^N)$. It clearly has the property that if π' is the orthogonal Koopman representation on $L^2(\mathbb{R}^N, \mu^N, \mathbb{R})$ induced by a_π, π' leaves $H^{:1:}$ invariant and $\pi' | H^{:1:}$ is (via the identification via θ) the orthogonal representation induced by π on $H_\mathbb{R}$. Clearly if $\pi \cong \rho$ (as unitary representations), then $a_\pi \cong a_\rho$ (as actions).

Next consider the unitary Koopman representation κ^{a_π} induced by a_π on $L^2(\mathbb{R}^N, \mu^N, \mathbb{C})$ (thus $\kappa^{a_\pi} = \pi' + i \cdot \pi'$). Let also $\kappa_0^{a_\pi}$ be its restriction to $L^2_0(\mathbb{R}^N, \mu^N, \mathbb{C})$. We claim that $\pi \leq \kappa_0^{a_\pi}$, i.e., π is isomorphic to a subrepresentation of $\kappa_0^{a_\pi}$. To see this, recall from Appendix A that there is a copy $\tilde{H}$ of H in $(H_\mathbb{R})_\mathbb{C}$, so that the copy $\tilde{T}$ of any $T \in U(H)$ is the restriction of $T + i \cdot T$ to $\tilde{H}$. It follows that $\theta_\mathbb{C}(\tilde{H}) \subseteq H^{:1:}_\mathbb{C} \subseteq L^2_0(\mathbb{R}^N, \mu^N, \mathbb{C})$ is invariant under $\kappa_0^{a_\pi}$ and that $\kappa_0^{a_\pi} | \theta_\mathbb{C}(\tilde{H}) \cong \pi$, so $\pi \leq \kappa_0^{a_\pi}$.

The discussion in this appendix provides a proof of the following result stated in Zimmer [Zi1], 5.2.13.

Theorem E.1. *Let* Γ *be a countable group and* π *a unitary representation of* Γ *on a (complex) Hilbert space* H*. Then one can explicitly construct a measure preserving action* a_π *of* Γ *on* (X, μ) *such that* $\pi \cong \rho \Rightarrow a_\pi \cong a_\rho$ *and if* $\kappa_0^{a_\pi}$ *is the unitary Koopman representation of* Γ *on* $L_0^2(X, \mu)$ *associated to* a_π*, then* $\pi \leq \kappa_0^{a_\pi}$.

If π has no non-0 finite-dimensional subrepresentations, i.e., it is weak mixing, then a_π is weak mixing. This is proved by an argument as in the proof of 11.2. See also Glasner [Gl2], 3.59.

APPENDIX F

Unitary representations of abelian groups

(A) Let Δ be a countable abelian group and denote by $\hat{\Delta}$ its dual group (where Δ is viewed as discrete). So $\hat{\Delta}$ is a compact Polish group. Denote by $\langle \delta, \hat{\delta} \rangle$ the duality between Δ and $\hat{\Delta}$, so that $\delta(\hat{\delta}) = \hat{\delta}(\delta) = \langle \delta, \hat{\delta} \rangle$, where we view $\delta \in \Delta$ as a character of $\hat{\Delta}$ and $\hat{\delta} \in \hat{\Delta}$ as a character of Δ. Let $\varphi : \Delta \to \mathbb{C}$ be a *positive-definite function*, i.e., for any finite $F \subseteq \Delta$ and $a_\gamma \in \mathbb{C}, \gamma \in F$, we have $\sum_{\gamma,\delta \in F} \overline{a}_\gamma a_\delta \varphi(\gamma^{-1}\delta) \geq 0$. Then, by *Bochner's Theorem* (see, e.g., Rudin [Ru2]), there is a unique finite (positive) Borel measure μ on $\hat{\Delta}$ such that for any $\delta \in \Delta$, if we put

$$\hat{\mu}(\delta) = \int \overline{\delta} d\mu,$$

then

$$\hat{\mu} = \varphi.$$

Conversely, for each such $\mu, \hat{\mu}$ is positive definite.

Let now $\pi \in \mathrm{Rep}(\Delta, H)$ be a unitary representation of Δ on a separable Hilbert space H. For each $h \in H$, let

$$\varphi_{\pi,h}(\delta) = \varphi_h(\delta) = \langle \pi(\delta)(h), h \rangle.$$

Then φ_h is positive-definite, so let $\mu_{\pi,h} = \mu_h$ be the corresponding measure on $\hat{\Delta}$:

$$\hat{\mu}_{\pi,h}(\delta) = \hat{\mu}_h(\delta) = \langle \pi(\delta)(h), h \rangle.$$

A vector $h_0 \in H$ is called a *cyclic vector* for π if H is the closed linear span of $\{\pi(\delta)(h_0) : \delta \in \Delta\}$. If π admits a cyclic vector h_0, then (H, π, h_0) is the GNS representation associated with φ_{h_0}. For each finite Borel measure μ on $\hat{\Delta}$ consider also the representation π_μ of Δ on $L^2(\hat{\Delta}, \mu)$ given by:

$$\pi_\mu(\delta)(f) = \overline{\delta} f.$$

Then $(L^2(\hat{\Delta}, \mu_{h_0}), \pi_{\mu_{h_0}}, 1)$ is also the GNS representation corresponding to φ_{h_0}, so there is an isomorphism of π with $\pi_{\mu_{h_0}}$ sending h_0 to 1.

Now recall that a *projection valued measure* (into H) on a standard Borel space X is a map $E : \mathrm{Bor}(X) \to \mathrm{Pr}(H)$ from the class of Borel subsets of X into the set of projections on H such that $E(\emptyset) = 0, E(X) = I, E(A \cap B) = E(A)E(B)$ and $E(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty E(A_i)$, if $\{A_i\}$ are pairwise disjoint.

We now claim that there is a projection valued measure $E : \mathrm{Bor}(\hat{\Delta}) \to \mathrm{Pr}(H)$ such that for any $h \in H$ and Borel set $A \subseteq \hat{\Delta}$,

$$\mu_h(A) = \langle E(A)(h), h\rangle = \|E(A)(h)\|^2.$$

Such an E is called a *spectral measure* for the representation π. Since we can decompose H into a direct sum of cyclic subrepresentations (i.e., subrepresentations admitting a cyclic vector), we can assume that π admits a cyclic vector h_0 and therefore we can assume that $H = L^2(\hat{\Delta}, \mu), h_0 = 1, \pi = \pi_\mu$, where $\mu = \mu_{h_0}$. So we need to define $E : \mathrm{Bor}(\hat{\Delta}) \to \mathrm{Pr}(L^2(\hat{\Delta}, \mu))$ such that $\mu_f(A) = \langle E(A)(f), f\rangle$, for each $f \in L^2(\hat{\Delta}, \mu)$. Indeed, let

$$E(A) = \chi_A L^2(\hat{\Delta}, \mu)$$

be the projection to the subspace of all $f \in L^2(\hat{\Delta}, \mu)$ supported by A. Then we need to check that $\mu_f(A) = \int \chi_A d\mu_f = \langle \chi_A f, f\rangle = \int \chi_A |f|^2 d\mu$. Now for $\delta \in \Delta, \int \overline{\delta} d\mu_f = \langle \pi_\mu(\delta)(f), f\rangle = \int \overline{\delta}|f|^2 d\mu$, so, using the fact that linear combinations of characters of $\hat{\Delta}$ are uniformly dense in $C(\hat{\Delta})$ (by Stone-Weierstrass) and using the Lebesgue Dominated Convergence Theorem, we see that for any bounded Borel $g : \hat{\Delta} \to \mathbb{C}$ we have $\int g d\mu_f = \int g|f|^2 d\mu$, so we are done putting $g = \chi_A$.

(B) The *maximal spectral type* for π is a finite Borel measure σ on $\hat{\Delta}$ such that for any Borel set $A \subseteq \hat{\Delta}$:

$$\begin{aligned} \sigma(A) = 0 \ &\Leftrightarrow \ E(A) = 0 \\ &\Leftrightarrow \ \forall h \in H(\mu_h(A) = 0). \end{aligned}$$

This is uniquely determined up to measure equivalence. Fix an orthonormal basis $\{e_n\}_{n=1}^\infty$ for H and put

$$\sigma_\pi = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_{e_n}.$$

Then clearly σ_π is the maximal spectral type of π and the map $\pi \mapsto \sigma_\pi$ is continuous from $\mathrm{Rep}(\Delta, H)$ (see Section 13) into $P(\hat{\Delta})$, the space of probability Borel measures on $\hat{\Delta}$ with the weak*-topology. Note that if h_0 is a cyclic vector for π, then μ_{h_0} is the maximal spectral type for π. To see this, it is enough to consider the representation $\pi = \pi_\mu$ on $L^2(\hat{\Delta}, \mu)$ and the cyclic vector 1, whose associated measure is μ. If $f \in L^2(\hat{\Delta}, \mu)$, then $\hat{\mu}_f(\delta) = \int (\overline{\delta} f)\overline{f} d\mu = \int \overline{\delta}|f|^2 d\mu = \hat{\nu}(\delta)$, where $d\nu = |f|^2 d\mu$. Thus $\mu_f = \nu \ll \mu$. It follows that for any π there is $h \in H$ with μ_h the maximal spectral type of π.

(C) Now let Δ be infinite and consider the Gaussian shift s on $\mathbb{R}^\Delta$ corresponding to $\varphi_1 =$ the characteristic function of $\{1\} \subseteq \Delta$. The corresponding Gaussian measure is the product measure μ^Δ, where μ is the normalized Gaussian measure on $\mathbb{R}$. Thus s is free and ergodic. We will next calculate that the maximal spectral type for κ_0^s is $\eta_{\hat{\Delta}} =$ Haar measure on $\hat{\Delta}$ (see Section 10, **(C)** for the definition of the Koopman representation κ_0^s corresponding to s). Using Appendix D, **(E)** and noticing the obvious fact that $\eta_{\hat{\Delta}}$ is the maximal spectral type of λ_Δ = the left-regular representation of Δ, we see that it is enough to show that if $F \leq \Delta$ is a finite subgroup and $\lambda_{\Delta/F}$ is the quasi-regular representation given by the action of Δ on $\ell^2(\Delta/F)$, then the maximal spectral type for $\lambda_{\Delta/F}$ is $\ll \eta_{\hat{\Delta}}$. Indeed, let $\Lambda \leq \hat{\Delta}$ be the intersection of the kernels of all $\delta \in F$. As F is finite, any such $\delta \in F$ satisfies $\delta^n = 1$, for some n, and therefore $\delta(\hat{\Delta})$ is finite, so $\ker(\delta)$ has finite index and thus is clopen. So Λ is clopen and its Haar measure is $\eta_\Lambda = \frac{\eta_{\hat{\Delta}}|\Lambda}{\eta_{\hat{\Delta}}(\Lambda)}$. The measure σ associated to the cyclic vector $\chi_{\{F\}} \in \ell^2(\Delta/F)$ of the representation $\lambda_{\Delta/F}$ is the maximal spectral type of $\lambda_{\Delta/F}$ and satisfies $\hat{\sigma}(\delta) = 1$, if $\delta \in F, \hat{\sigma}(\delta) = 0$, if $\delta \notin F$. Clearly $\hat{\eta}_\Lambda(\delta) = 1$, if $\delta \in F$. If $\delta \notin F$, then by Rudin [Ru2], proof of 2.1.3, there is $\hat{\delta} \in \Lambda$ such that $\delta(\hat{\delta}) \neq 1$. So $\delta|\Lambda$ is a non-trivial character of Λ, thus $\hat{\eta}_\Lambda(\delta) = 0$. So $\hat{\sigma}(\delta) = \hat{\eta}_\Lambda(\delta)$, for all δ, and thus $\sigma = \eta_\Lambda \ll \eta_{\hat{\Delta}}$.

APPENDIX G

Induced representations and actions

We will review here some facts about induced representations and actions. Our general reference is Bekka-de la Harpe-Valette [BdlHV], Part II.

(A) Let $\Gamma \leq \Delta$ be countable groups. Fix a transversal T for the left cosets of Γ in Δ containing $1 \in \Gamma$. Then Δ acts on T by

$$\delta \cdot t = \text{the unique element of } T \text{ in the coset } \delta t\Gamma,$$

and we also define the following cocycle for this action, $\rho : \Delta \times T \to \Gamma$,

$$\rho(\delta, t) = \gamma, \text{ where } \delta t = (\delta \cdot t)\gamma.$$

Let now $\pi \in \mathrm{Rep}(\Gamma, H)$ be a unitary representation of Γ on H. Consider the direct sum $\bigoplus_{t \in T} H$ of $\mathrm{card}(T) = |T|$ copies of H. Define then the *induced representation*

$$\mathrm{Ind}_\Gamma^\Delta(\pi) \in \mathrm{Rep}(\Delta, \bigoplus_{t \in T} H)$$

by

$$\delta \cdot (\bigoplus_{t \in T} h_t) = \bigoplus_{t \in T} \rho(\delta^{-1}, t)^{-1} \cdot h_{\delta^{-1} \cdot t}.$$

It is easy to check, using the cocycle property for ρ, that this is indeed a unitary representation.

Let now $a \in A(\Gamma, X, \mu)$, and assume that $[\Delta : \Gamma] < \infty$, i.e., T is finite. Denote the normalized counting measure on T by ν_T. Consider the space $X \times T$ with measure $\mu \times \nu_T$ and define the *induced action*

$$\mathrm{Ind}_\Gamma^\Delta(a) \in A(\Delta, X \times T, \mu \times \nu_T)$$

by

$$\delta \cdot (x, t) = (\rho(\delta, t) \cdot x, \delta \cdot t).$$

It is easy to check that this is indeed measure preserving. If a is free (resp., ergodic), then $\mathrm{Ind}_\Gamma^\Delta(a)$ is free (resp., ergodic).

We next verify that for $[\Delta : \Gamma] < \infty$ and $a \in A(\Gamma, X, \mu)$,

$$\kappa^{\mathrm{Ind}_\Gamma^\Delta(a)} \cong \mathrm{Ind}_\Gamma^\Delta(\kappa^a),$$

i.e., induction commutes with the Koopman representation. To see this let $f \in L^2(X \times T, \mu \times \nu_T) =$ the Hilbert space of $\kappa^{\mathrm{Ind}_\Gamma^\Delta(a)}$, and define $U(f) \in$

$\bigoplus_{t\in T} L^2(X,\mu)=$ the Hilbert space of $\mathrm{Int}_\Gamma^\Delta(\kappa^a)$, by

$$U(f)=\frac{1}{\sqrt{|T|}}\bigoplus_{t\in T} f_t,$$

where $f_t(x)=f(x,t)$. Then it can be easily checked that U is an isomorphism of the Hilbert spaces $L^2(X\times T,\mu\times\nu_T)$ and $\bigoplus_{t\in T} L^2(X,\mu)$. We finally verify that it preserves the Δ-actions (corresponding to $\kappa^{\mathrm{Ind}_\Gamma^\Delta(a)}, \mathrm{Ind}_\Gamma^\Delta(\kappa^a)$, resp.). Indeed, we have

$$\begin{aligned} U(\delta\cdot f) &= U((x,t)\mapsto f(\delta^{-1}\cdot(x,t)))\\ &= U((x,t)\mapsto f(\rho(\delta^{-1},t)\cdot x,\delta^{-1}\cdot t)))\\ &= \tfrac{1}{\sqrt{|T|}}\textstyle\bigoplus_t (x\mapsto f(\rho(\delta^{-1},t)\cdot x,\delta^{-1}\cdot t)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta\cdot U(f) &= \delta\cdot\tfrac{1}{\sqrt{|T|}}\textstyle\bigoplus_t f_t\\ &= \tfrac{1}{\sqrt{|T|}}\textstyle\bigoplus_t \rho(\delta^{-1},t)^{-1}\cdot f_{\delta^{-1}\cdot t}\\ &= \tfrac{1}{\sqrt{|T|}}\textstyle\bigoplus_t (x\mapsto f_{\delta^{-1}\cdot t}(\rho(\delta^{-1},t)\cdot x))\\ &= \tfrac{1}{\sqrt{|T|}}\textstyle\bigoplus_t (x\mapsto f(\rho(\delta^{-1},t)\cdot x,\delta^{-1}\cdot t)), \end{aligned}$$

so $U(\delta\cdot f)=\delta\cdot U(f)$.

It follows that if $a\in A(\Gamma,X,\mu), b\in A(\Gamma,Y,\nu)$ are unitarily equivalent, i.e., $\kappa^a\cong\kappa^b$, then so are $\mathrm{Ind}_\Gamma^\Delta(a)$, $\mathrm{Ind}_\Gamma^\Delta(b)$.

(B) Now assume that Δ is abelian with dual group $\hat{\Delta}$. We use below the notation of Appendix F. If θ is an automorphism of Δ, then we define the automorphism of $\hat{\theta}$ of $\hat{\Delta}$ by $\langle\delta,\hat{\theta}(\hat{\delta})\rangle=\langle\theta(\delta),\hat{\delta}\rangle$.

For each representation $\pi\in\mathrm{Rep}(\Delta,H)$ and automorphism θ of Δ, we have the corresponding representation

$$\theta(\pi)=\pi\circ\theta\in\mathrm{Rep}(\Delta,H).$$

For each $\delta\in\Delta, h\in H, \hat{\mu}_{\theta(\pi),h}(\delta)=\langle\theta(\pi)(\delta)(h),h\rangle=\langle\pi(\theta(\delta))(h),h\rangle=\varphi_{\pi,h}(\theta(\delta))=\hat{\mu}_{\pi,h}(\theta(\delta))$, so $\mu_{\theta(\pi),h}=(\hat{\theta})_*\mu_{\pi,h}$ and therefore if σ is the maximal spectral type for π, $(\hat{\theta})_*\sigma$ is the maximal spectral type for $\theta(\pi)$.

Let now $\Delta\lhd\Gamma$ with $[\Gamma:\Delta]<\infty$, and consider $\pi\in\mathrm{Rep}(\Delta,H)$ and the induced representation $\mathrm{Ind}_\Delta^\Gamma(\pi)\in\mathrm{Rep}(\Gamma,\bigoplus_{t\in T}H)$. Let $\overline{\pi}=\mathrm{Ind}_\Delta^\Gamma(\pi)|\Delta\in\mathrm{Rep}(\Delta,\bigoplus_{t\in T}H)$. Then for the representation $\overline{\pi}$ we have

$$\begin{aligned} \delta\cdot(\textstyle\bigoplus_{t\in T}h_t) &= \textstyle\bigoplus_{t\in T}\rho(\delta^{-1},t)^{-1}\cdot h_{\delta^{-1}\cdot t}\\ &= \textstyle\bigoplus_{t\in T}(t^{-1}\delta t)\cdot h_t,\\ &= \textstyle\bigoplus_{t\in T}\theta_t(\delta)\cdot h_t, \end{aligned}$$

where $\theta_t(\delta)=t^{-1}\delta t$. Thus

$$\bar{\pi}=\bigoplus_{t\in T}\theta_t(\pi).$$

It follows that if σ is the maximal spectral type for π, then

$$\sum_{t\in T}(\hat{\theta}_t)_*\sigma$$

is the maximal spectral type for $\bar{\pi} = \mathrm{Ind}_\Delta^\Gamma(\pi)|\Delta$.

(I would like to thank Dinakar Ramakrishnan for a useful discussion here.)

APPENDIX H

The space of unitary representations

(A) We use here Bekka-de la Harpe-Valette [BdlHV] as a general reference for unitary representations.

Let H be a separable complex Hilbert space. We endow $U(H)$, the unitary group of H, with the *strong topology*, i.e., the one generated by the family of maps $T \in U(H) \mapsto T(x) \in H, x \in H$. This is the same as the *weak topology*, generated by the maps $T \in U(H) \mapsto \langle T(x), y\rangle \in \mathbb{C}, x, y \in H$. Then $U(H)$ is a Polish group.

If now Γ is a countable group, we let $\mathrm{Rep}(\Gamma, H)$ be the set of all homomorphisms for Γ into $U(H)$, i.e., the set of unitary representations of Γ on H. We also write $H = H_\pi$ if $\pi \in \mathrm{Rep}(\Gamma, H)$. If we give $U(H)^\Gamma$ the product topology, so it becomes a Polish space, then $\mathrm{Rep}(\Gamma, H)$ is a closed subspace of $U(H)^\Gamma$, thus it is Polish too. If $\{e_m\}$ is an orthonormal basis for H, then an open nbhd basis for $\pi \in \mathrm{Rep}(\Gamma, H)$ consists of the sets of the form

$$\{\sigma \in \mathrm{Rep}(\Gamma, H) : \forall i, j \leq k \forall \gamma \in F(|\langle\sigma(\gamma)(e_i), e_j\rangle - \langle\pi(\gamma)(e_i), e_j\rangle| < \epsilon)\},$$

where $k = 1, 2, \ldots, F \subseteq \Gamma$ is finite and $\epsilon > 0$.

The group $U(H)$ acts continuously on $\mathrm{Rep}(\Gamma, H)$ by conjugation, i.e.,

$$T \cdot \pi = T\pi T^{-1},$$

where

$$T\pi T^{-1}(\gamma) = T\pi(\gamma)T^{-1}.$$

Let us note here that if F_N is the free group with N generators, $N \in \{1, 2, \ldots, \mathbb{N}\}$ and $\alpha_1, \alpha_2, \ldots$ are free generators, the map $\pi \in \mathrm{Rep}(F_N, H) \mapsto (\pi(\alpha_1), \pi(\alpha_2), \ldots) \in U(H)^N$ is a homeomorphism of $\mathrm{Rep}(F_N, H)$ with the space $U(H)^N$ preserving conjugation. Thus we can identify $\mathrm{Rep}(F_N, H)$ with $U(H)^N$. In particular, $\mathrm{Rep}(\mathbb{Z}, H)$ can be identified with $U(H)$.

The representations $\pi \in \mathrm{Rep}(\Gamma, H_\pi), \rho \in \mathrm{Rep}(\Gamma, H_\rho)$ are *isomorphic*, in symbols

$$\pi \cong \rho,$$

if there is an isomorphism T of H_π, H_ρ with $T\pi(\gamma)T^{-1} = \rho(\gamma), \forall\gamma \in \Gamma$. If we consider the conjugacy action of $U(H)$ on $\mathrm{Rep}(\Gamma, H)$, then clearly for $\pi, \rho \in \mathrm{Rep}(\Gamma, H), \pi \cong \rho$ iff π, ρ are conjugate.

We have the following easy fact.

Proposition H.1. *Let Γ be a countable group. If H is infinite-dimensional, there is a dense conjugacy class in* $\mathrm{Rep}(\Gamma, H)$.

Proof. Let $\{\pi_n\}$ be dense in $\mathrm{Rep}(\Gamma, H)$. Then it is easy to check that $\{\pi \in \mathrm{Rep}(\Gamma, H) : \pi \cong \bigoplus_n \pi_n\}$ is dense in $\mathrm{Rep}(\Gamma, H)$. Indeed fix a nbhd

$$V = \{\sigma : \forall i, j \leq k \forall \gamma \in F(|\langle \sigma(\gamma)(e_i), e_j\rangle - \langle \pi_{n_0}(\gamma)(e_i), e_j\rangle| < \epsilon)\}$$

of π_{n_0}, where $\{e_m\}$ is an orthonormal basis for $H, k = 1, 2, \dots, F \subseteq \Gamma$ is finite and $\epsilon > 0$. Now $\bigoplus_n \pi_n \in \mathrm{Rep}(\Gamma, K)$, where $K = \bigoplus_n H_n$, with H_n isomorphic to H. Fix an orthonormal basis $\{e_m^{(n)}\}$ for each H_n so that $T(e_m^{(n_0)}) = e_m$ is an isomorphism of H_{n_0} with H that sends $\bigoplus_n \pi_n | H_{n_0}$ to π_{n_0}. Consider then an isomorphism $S : K \to H$ such that $S(e_i^{(n_0)}) = e_i$ for $i \leq k$, and let $\sigma \in \mathrm{Rep}(\Gamma, H)$ be the image of $\bigoplus_n \pi_n$ under S. Then clearly $\sigma \in V$. ☐

(B) If π is *contained* in ρ, i.e., π is isomorphic to a subrepresentation (i.e., restriction to a closed invariant subspace) of ρ, we write

$$\pi \leq \rho.$$

We say that π is *weakly contained* in ρ, in symbols

$$\pi \prec \rho,$$

if every positive-definite function realized in π is the pointwise limit of a sequence of finite sums of positive definite functions realized in ρ. Recalling that a positive-definite function *realized* in a representation $\sigma \in \mathrm{Rep}(\Gamma, H)$ is a function $f : \Gamma \to \mathbb{C}$ given by $f(\gamma) = \langle \sigma(\gamma)(v), v\rangle$, for some $v \in H$, this means that for any $v \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there are $v_1, \dots, v_k \in H_\rho$ such that $|\langle \pi(\gamma)(v), v\rangle - \sum_{i=1}^k \langle \rho(\gamma)(v_i), v_i\rangle| < \epsilon, \forall \gamma \in F$.

Clearly $\prec$ is transitive. We say that π, ρ are *weakly equivalent*, in symbols,

$$\pi \sim \rho,$$

if $\pi \prec \rho$ and $\rho \prec \pi$. Clearly $\pi \leq \rho \Rightarrow \pi \prec \rho, \pi \cong \rho \Rightarrow \pi \sim \rho$. Note that

$$\pi \sim n \cdot \pi$$

for any $n = 1, 2, \dots, \infty$, where $n \cdot \pi$ is the direct sum of n copies of π (and ∞ represents $\aleph_0$). One can then show (see, e.g., Kechris [Kec4], 2.2) that $\pi \prec \rho$ is equivalent to any of the following:

(i) Every positive definite function realized in π is the pointwise limit of a sequence of positive definite functions realized in $\infty \cdot \rho$.

(ii) For every $v_1, \dots, v_k \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there are $w_1, \dots, w_k \in H_{\infty \cdot \rho}$ such that $|\langle \pi(\gamma)(v_i), v_j\rangle - \langle \infty \cdot \rho(\gamma)(w_i), w_j\rangle| < \epsilon, \forall \gamma \in F, i, j \leq k$.

(iii) Same as (ii) for $v_1, \dots, v_k$ orthonormal and requiring that $w_1, \dots, w_k$ are also orthonormal.

We can use (ii), (iii) to show the following fact.

Proposition H.2. *Let $\pi \in \mathrm{Rep}(\Gamma, H_\pi), \rho \in \mathrm{Rep}(\Gamma, H_\rho)$, where H_π, H_ρ are infinite-dimensional. Then*

$$\pi \prec \rho \Leftrightarrow \pi \in \overline{\{\sigma \in \mathrm{Rep}(\Gamma, H_\pi) : \sigma \cong \infty \cdot \rho\}}.$$

Proof. $\Rightarrow$: Fix an orthonormal basis $\{e_n\}$ for H_π, and a basic nbhd

$$V = \{\sigma : \forall i, j \leq k \forall \gamma \in F | \langle \sigma(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(e_i), e_j \rangle | < \epsilon\},$$

($k = 1, 2, \ldots, F \subseteq \Gamma$ finite, $\epsilon > 0$) of π. If $\pi \prec \rho$, then there is an orthonormal set $w_1, \ldots, w_k$ in $H_{\infty \cdot \rho}$ such that $|\langle \pi(\gamma)(e_i), e_j \rangle - \langle \infty \cdot \rho(\gamma)(w_i), w_j \rangle| < \epsilon, \forall i, j \leq k, \forall \gamma \in F$. Fix then an isomorphism $T : H_\pi \to H_{\infty \cdot \rho}$ with $T(e_i) = w_i, \forall i \leq k$, and let σ be the copy of $\infty \cdot \rho$ in H_π induced by T^{-1}. Then $\langle \sigma(\gamma)(e_i), e_j \rangle = \langle \infty \cdot \rho(\gamma)(w_i), w_j \rangle$, so $\sigma \in V$, thus $V \cap \{\sigma \in \text{Rep}(\Gamma, H_\pi) : \sigma \cong \infty \cdot \rho\} \neq \emptyset$.

$\Leftarrow$: Given $v_1, \ldots, v_k \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite,

$$\{\sigma \in \text{Rep}(\Gamma, H_\pi) : \forall i, j \leq k \forall \gamma \in F | \langle \sigma(\gamma)(v_i), v_j \rangle - \langle \pi(\gamma)(v_i), v_j \rangle | < \epsilon\}$$

is an open nbhd of π, so it contains $\sigma \cong \infty \cdot \rho$. It is clear then that there are $w_1, \ldots, w_k \in H_{\infty \cdot \rho}$ with $|\langle \pi(\gamma)(v_i), v_j \rangle - \langle \infty \cdot \rho(\gamma)(w_i), w_j \rangle| < \epsilon$, so $\pi \prec \infty \cdot \rho$. □

Thus $\pi \prec \rho$ on $\text{Rep}(\Gamma, H)$, H infinite-dimensional, is, except for replacing ρ by $\infty \cdot \rho$, the partial preordering associated with the conjugacy action of $U(H)$ on $\text{Rep}(\Gamma, H)$ (see the paragraph preceding 10.3).

Remark. Using the terminology of Bekka-de la Harpe-Valette [BdlHV], F.1.2, one says that π is *weakly contained in the sense of Zimmer* in ρ, in symbols $\pi \prec_Z \rho$, if for every $v_1, \ldots, v_n \in H_\pi, \epsilon > 0, F \subseteq \Gamma$ finite, there are $w_1, \ldots, w_n \in H_\rho$ such that $|\langle \pi(\gamma)(v_i), v_j \rangle - \langle \rho(\gamma)(w_i), w_j \rangle| < \epsilon, \forall \gamma \in F, i, j \leq n$. This means that (ii) above holds for ρ instead of $\infty \cdot \rho$. It is easy to show (see, e.g., Kechris [Kec4], 2.2) that this is equivalent again to this condition for $v_1, \ldots, v_n$ orthonormal and requiring that $w_1, \ldots, w_n$ are also orthonormal. It follows, as in proposition H.2, that for $\pi \in \text{Rep}(\Gamma, H_\pi), \rho \in \text{Rep}(\Gamma, H_\rho)$, where H_π, H_ρ are infinite-dimensional,

$$\pi \prec_Z \rho \Leftrightarrow \pi \in \overline{\{\sigma \in \text{Rep}(\Gamma, H_\pi) : \sigma \cong \rho\}}.$$

Thus $\prec_Z$ corresponds exactly to the partial preordering associated with the conjugacy action of $U(H)$ on $\text{Rep}(\Gamma, H)$. We also have that $\pi \prec_Z \rho \Rightarrow \pi \prec \rho$ and $\pi \prec \rho \Leftrightarrow \pi \prec_Z \infty \cdot \rho$. As Bekka-de la Harpe-Valette [BdlHV], F.1.2 point out, if π is irreducible, then $\pi \prec \rho \Leftrightarrow \pi \prec_Z \rho$. In particular, $1_\Gamma \prec \pi \Leftrightarrow 1_\Gamma \prec_Z \pi \Leftrightarrow \pi$ admits non-0 almost invariant vectors.

Corollary H.3. *For H infinite-dimensional, the relation $\pi \prec \rho$ on the space* $\text{Rep}(\Gamma, H)$ *is a G_δ subset of* $\text{Rep}(\Gamma, H)^2$. *Every section $\prec_\rho = \{\pi : \pi \prec \rho\}$ is closed in* $\text{Rep}(\Gamma, H)$.

Proof. The first assertion follows from the following easy lemma.

Lemma H.4. *Let $\pi \in \text{Rep}(\Gamma, H_\pi)$ and $\rho \in \text{Rep}(\Gamma, H_\rho)$. Fix a countable dense set $D \subseteq H_\pi$. Then $\pi \prec \rho$ iff any positive-definite function of the form $\gamma \mapsto \langle \pi(\gamma)(v), v \rangle$, for $v \in D$, is the pointwise limit of a sequence of finite sums of positive-definite functions realized in ρ.*

The second assertion follows from Proposition H.2. □

(C) A representation $\pi \in \mathrm{Rep}(\Gamma, H)$ is *irreducible* if it admits no non-trivial subrepresentation, i.e., the only invariant closed subspaces are $\{0\}, H$. We denote by

$$\mathrm{Irr}(\Gamma, H)$$

the set of irreducible representations in $\mathrm{Rep}(\Gamma, H)$.

Proposition H.5. $\mathrm{Irr}(\Gamma, H)$ *is* G_δ *in* $\mathrm{Rep}(\Gamma, H)$ *(and thus a Polish space in the relative topology).*

Proof. Let $\pi \in \mathrm{Rep}(\Gamma, H)$. Fix a unit vector $h \in H$ and let $\varphi_{\pi,h}(\gamma) = \varphi_h(\gamma) = \langle \pi(\gamma)(h), h\rangle$ be the corresponding positive-definite function. Then (see Bekka-de la Harpe-Valette [BdlHV], C.5.2) π is irreducible iff h is a cyclic vector for π and φ_h is an extreme point in the compact convex set $\mathcal{P}_1(\Gamma)$ of all positive-definite functions $\varphi : \Gamma \to \mathbb{C}$ with $\varphi(1) = 1$. Recall that $h \in H$ is cyclic if the closed linear span of $\{\pi(\gamma)(h) : \gamma \in \Gamma\}$ is equal to H. We view here $\mathcal{P}_1(\Gamma)$ as a compact convex subset of the unit ball of $\ell^\infty(\Gamma) = \ell^\infty(\Gamma, \mathbb{C})$ with the weak*-topology (i.e., pointwise convergence topology). It is clear that the set $\mathrm{ext}(\mathcal{P}_1(\Gamma))$ of extreme points of $\mathcal{P}_1(\Gamma)$ is G_δ in $\mathcal{P}_1(\Gamma)$ and that $\pi \mapsto \varphi_{\pi,h}$ is continuous from $\mathrm{Rep}(\Gamma, H)$ into $\mathcal{P}_1(\Gamma)$. Thus

$$\pi \in \mathrm{Irr}(\Gamma, H) \Leftrightarrow h \text{ is cyclic for } \pi \text{ and } \varphi_{\pi,h} \in \mathrm{ext}(\mathcal{P}_1(\Gamma))$$

is G_δ in $\mathrm{Rep}(\Gamma, H)$. □

The following result is well-known (see, for example, [Ef2], III or [Hj3]).

Theorem H.6. *Isomorphism on* $\mathrm{Irr}(\Gamma, H)$ *is an* F_σ *equivalence relation* (*as a subset of* $\mathrm{Irr}(\Gamma, H)^2$).

Proof. We use *Schur's Lemma* (see Folland [Fo], 3.5 (b)), which asserts that $\pi_1, \pi_2 \in \mathrm{Irr}(\Gamma, H)$ are isomorphic iff there is a non-0 bounded linear operator S in H such that $\forall \gamma \in \Gamma(S\pi_1(\gamma) = \pi_2(\gamma)S)$ (such a S is called an *intertwining operator* for π_1, π_2). Denote by $B_1(H)$ the set of bounded linear operators on H of norm ≤ 1 with the weak operator topology, i.e., the one induced by the maps $T \mapsto \langle Tx, y\rangle$, for $x, y \in H$. Then $B_1(H)$ is compact metrizable, by Alaoglu's Theorem and Sunder [Sun], 0.3.1 (b), and $U(H)$ is a G_δ subset of $B_1(H)$. So we have for $\pi_1, \pi_2 \in \mathrm{Irr}(\Gamma, H)$,

$$\pi_1 \cong \pi_2 \Leftrightarrow \exists S \in B_1(H)[S \neq 0 \text{ and } \forall \gamma \in \Gamma(S\pi_1(\gamma) = \pi_2(\gamma)S)].$$

Claim. *There is compact* $K \subseteq B_1(H)^3$ *such that* $K \cap (U(H)^2 \times B_1(H)) = \{(T_1, T_2, S) : ST_1 = T_2 S\}$.

Granting this, we have for $\pi_1, \pi_2 \in \mathrm{Irr}(\Gamma, H)$,

$$\pi_1 \cong \pi_2 \Leftrightarrow \exists S \in B_1(H)[S \neq 0 \text{ and } \forall \gamma \in \Gamma(\pi_1(\gamma), \pi_2(\gamma), S) \in K)].$$

Now the topology on $\mathrm{Irr}(\Gamma, H) \subseteq \mathrm{Rep}(\Gamma, H) \subseteq U(H)^\Gamma \subseteq B_1(H)^\Gamma$ is the relative topology from $B_1(H)^\Gamma$, which is a compact metrizable space, and

$$R = \{(\pi_1, \pi_2, S) \in (B_1(H)^\Gamma)^2 \times B_1(H) : \forall \gamma (\pi_1(\gamma), \pi_2(\gamma), S) \in K\}$$

is compact, so

$$Q = \{(\pi_1, \pi_2) \in (B_1(H)^\Gamma)^2 : \exists S \in B_1(H)[S \neq 0 \text{ and } (\pi_1, \pi_2, S) \in R]\}$$

is K_σ in $(B_1(H)^\Gamma)^2$, and

$$Q \cap \mathrm{Irr}(\Gamma, H)^2 = (\cong |\mathrm{Irr}(\Gamma, H)^2)$$

is F_σ in $\mathrm{Irr}(\Gamma, H)^2$.

Proof of the claim. We need to check that

$$\{(T_1, T_2, S) : ST_1 = T_2 S\}$$

is closed in $U(H)^2 \times B_1(H)$. Notice that

$$ST_1 = T_2 S \Leftrightarrow \forall x, y \in H[\langle ST_1(x), y\rangle = \langle x, S^* T_2^*(y)\rangle]$$

and, as $U \mapsto U^*$ is continuous in $B_1(H)$ (see Sunder [Sun], 0.3.2 (a)), it is enough to check that $(S, T) \mapsto \langle ST(x), y\rangle$ is continuous in $B_1(H) \times U(H), \forall x, y \in H$. Assume then that $S_n \to S, T_n \to T$ weakly, where $S_n, S \in B_1(H), T_n, T \in U(H)$. Since the weak and strong operator topologies agree on $U(H), T_n \to T$ strongly as well, so

$$\begin{aligned} |\langle S_n T_n(x), y\rangle - \langle ST(x), y\rangle| &\leq |\langle S_n T_n(x), y\rangle - \langle S_n T(x), y\rangle| \\ &\quad + |\langle S_n T(x), y\rangle - \langle ST(x), y\rangle| \\ &= |\langle S_n(T_n(x) - T(x)), y\rangle| \\ &\quad + |\langle S_n(z), y\rangle - \langle S(z), y\rangle|, \end{aligned}$$

where $z = T(x)$, and this is less than or equal to

$$\|S_n\| \cdot \|T_n(x) - T(x)\| \cdot \|y\| + |\langle S_n(z), y\rangle - \langle S(z), y\rangle|$$

which converges to 0 as $n \to \infty$, since $T_n \to T$ strongly and $S_n \to S$ weakly. □

Remark. Concerning the descriptive complexity of the relation of isomorphism on $\mathrm{Rep}(\Gamma, H)$, it is clearly Borel if Γ is abelian by the Spectral Theorem, but I do not know what is the situation for arbitrary Γ.

There are cases where $\mathrm{Irr}(\Gamma, H)$ is a dense G_δ.

Proposition H.7 (Hjorth [Hj2], 5.6, for $N = \infty$). *For each $N \geq 2$, the set $\mathrm{Irr}(F_N, H)$ is dense G_δ in $\mathrm{Rep}(F_N, H)$.*

Proof. By Section 4, **(D)**, the set of $(g_1, \ldots, g_N) \in U(H)^N$ such that $\langle g_1, \ldots, g_N\rangle$ is a dense subgroup of $U(H)$ is dense G_δ in $U(H)^N$. As $\mathrm{Rep}(F_N, H)$ can be identified with $U(H)^N$, it follows that $D = \{\pi \in \mathrm{Rep}(F_N, H) : \pi(F_n) \text{ is dense in } U(H)\}$ is dense G_δ in $\mathrm{Rep}(F_N, H)$, and clearly $D \subseteq \mathrm{Irr}(F_N, H)$. □

On the other hand we will see below, in H.13, that if Γ is infinite and has property (T) and H is infinite-dimensional, then $\mathrm{Irr}(\Gamma, H)$ is nowhere dense.

Thoma [Th] has shown that if Γ is not abelian-by-finite and H is infinite-dimensional, then $\cong |\mathrm{Irr}(\Gamma, H)$ is not smooth, and Hjorth [Hj3] extended this by showing that it cannot be classified by countable structures. The arguments in Hjorth [Hj2] in fact show that the action of $U(H)$ on the space $\mathrm{Irr}(F_N, H), N \geq 2$, is generically turbulent. (Thoma [Th] has also shown that when Γ is abelian-by-finite and H is infinite-dimensional, then $\mathrm{Irr}(\Gamma, H) = \emptyset$.)

We will now give a more detailed version of Hjorth's result. First let us note the following general fact. Recall that a subset $A \subseteq X$ of a topological space X is called *locally closed* if it is open in its closure.

Proposition H.8. *Let a Polish group G act continuously on a Polish space X and assume the equivalence relation E_G^X induced by the action is F_σ (as a subset of X^2). Then the set*

$$\mathrm{LC} = \{x \in X : G \cdot x \textit{ is locally closed}\}$$

is F_σ.

Proof. Fix an open basis $\{V_n\}$ for X. Let $E_G^X = \bigcup_n F_n, F_n$ closed in X^2. For $F \subseteq X^2, x \in X$, let $F(x) = \{y : (x, y) \in F\}$. Then

$$\begin{aligned} x \in \mathrm{LC} &\Leftrightarrow G \cdot x \text{ is open in } \overline{G \cdot x} \\ &\Leftrightarrow \bigcup_n F_n(x) \text{ is not meager in } \overline{G \cdot x} \\ &\Leftrightarrow \exists n \exists m (V_n \cap G \cdot x \neq \emptyset \ \& \ V_n \cap \overline{G \cdot x} \subseteq F_m(x)) \\ &\Leftrightarrow \exists n \exists m (x \in G \cdot V_n \ \& \ \forall g \in G(g \cdot x \in V_n \Rightarrow (x, g \cdot x) \in F_m)) \end{aligned}$$

which clearly shows that LC is F_σ. □

Consider now the conjugacy action of $U(H), H$ an infinite-dimensional, separable Hilbert space, on $\mathrm{Irr}(\Gamma, H)$. The corresponding equivalence relation is $\cong$ on $\mathrm{Irr}(\Gamma, H)$, which, by H.6, is F_σ. Thus the set $\mathrm{LC}(\Gamma, H)$ of irreducible representations with locally closed conjugacy class in $\mathrm{Irr}(\Gamma, H)$ is F_σ in $\mathrm{Irr}(\Gamma, H)$. Effros' theorem implies that $\cong |\mathrm{Irr}(\Gamma, H)$ is smooth iff $\mathrm{LC}(\Gamma, H) = \mathrm{Irr}(\Gamma, H)$, thus, by Thoma [Th], if Γ is not abelian-by-finite the set

$$\mathrm{NLC}(\Gamma, H) = \mathrm{Irr}(\Gamma, H) \setminus \mathrm{LC}(\Gamma, H)$$

of irreducible representations with non-locally closed conjugacy class in the space $\mathrm{Irr}(\Gamma, H)$ is a non-empty G_δ set and $\cong |\mathrm{NLC}(\Gamma, H)$ is also non-smooth (since $\cong |\mathrm{LC}(\Gamma, H)$ is easily smooth). I do not know if $\mathrm{NLC}(\Gamma, H)$ is dense in $\mathrm{Irr}(\Gamma, H)$.

Finally for each $\pi \in \mathrm{Irr}(\Gamma, H)$, let

$$S(\pi) = \{\sigma \in \mathrm{Irr}(\Gamma, H) : \sigma \prec \pi\} = \{\sigma \in \mathrm{Irr}(\Gamma, H) : \sigma \in \overline{U(H) \cdot \pi}\},$$

the latter equality following from the Remark after H.2. Clearly $S(\pi)$ is a conjugacy invariant closed subset of $\mathrm{Irr}(\Gamma, H)$. The following is a more detailed version of the main result in Hjorth [Hj3].

Theorem H.9 (Hjorth [Hj3]). *Let* Γ *be a countable group and assume* H *is infinite-dimensional. For each* $\pi \in \mathrm{NLC}(\Gamma, H)$*, the conjugacy action of* $U(H)$ *on* $S(\pi)$ *is generically turbulent. In particular, if* Γ *is not abelian-by-finite, then* $\cong |\mathrm{Irr}(\Gamma, H)$ *is not classifiable by countable structures.*

Proof. We will make use of the following standard lemma.

Lemma H.10. *Let* $\rho, \sigma \in \mathrm{Irr}(\Gamma, H)$*. Then the following are equivalent:*

(i) $\rho \cong \sigma$*,*

(ii) There is a sequence $\{T_n\}$ *in* $U(H)$ *such that* $T_n \cdot \rho = T_n \rho T_n^{-1} \to \sigma$ *and no subsequence of* $\{T_n\}$ *converges in the weak topology of* $B_1(H)$ *to* 0*.*

Proof. (i) $\Rightarrow$ (ii) is clear, as we can take $\{T_n\}$ to be a constant sequence.

(ii) $\Rightarrow$ (i): As $U(H) \subseteq B_1(H)$ and the weak topology on $B_1(H)$ is compact metrizable, by going to a subsequence, we can assume that $T_n \to^w T \in B_1(H)$, where $T \neq 0$. We claim now that T is an intertwining operator for ρ, σ and so, by Schur's Lemma, $\rho \cong \sigma$. Since $T \mapsto T^*$ is continuous in the weak topology, note that we also have $T_n^* = T_n^{-1} \to^w T^*$.

We need to show that

$$T\rho(\gamma^{-1}) = \sigma(\gamma^{-1})T, \forall \gamma \in \Gamma$$

or equivalently

$$\rho(\gamma)T^* = T^*\sigma(\gamma),$$

i.e.,

$$\langle \rho(\gamma)T^*(x), y\rangle = \langle T^*\sigma(\gamma)(x), y\rangle,$$

for $\gamma \in \Gamma, x, y \in H$.

Now as $T_n\rho(\gamma)T_n^{-1} \to \sigma(\gamma)$ in the topology of $U(H)$, clearly

$$\|T_n\rho(\gamma)T_n^{-1}(x) - \sigma(\gamma)(x)\| \to 0,$$

so $\|\rho(\gamma)T_n^{-1}(x) - T_n^{-1}\sigma(\gamma)(x)\| \to 0$, thus

$$|\langle \rho(\gamma)T_n^{-1}(x), y\rangle - \langle T_n^{-1}\sigma(\gamma)(x), y\rangle| \to 0.$$

But

$$\langle \rho(\gamma)T_n^{-1}(x), y\rangle = \langle T_n^{-1}(x), \rho(\gamma)^{-1}(y)\rangle \to \langle T^*(x), \rho(\gamma)^{-1}(y)\rangle = \langle \rho(\gamma)T^*(x), y\rangle$$

and $\langle T_n^{-1}\sigma(\gamma)(x), y\rangle \to \langle T^*\sigma(\gamma)(x), y\rangle$, so we are done. □

We thus have that if $\sigma \in S(\rho)$ (i.e., $\sigma \in \overline{U(H) \cdot \rho}$) and $\rho \not\cong \sigma$, then there is $\{T_n\}$ in $U(H)$ with $T_n \cdot \rho \to \sigma$ and $T_n \to^w 0$.

We proceed to verify that the conjugacy action of $U(H)$ on $S(\pi)$ is generically turbulent for the given $\pi \in \mathrm{NLC}(\Gamma, H)$. First it is clear that the conjugacy class of π is dense and meager in $S(\pi)$ (else it would be open in $S(\pi)$ contradicting that $\pi \in \mathrm{NLC}(\Gamma, H)$). If $\rho \in S(\pi) \setminus U(H) \cdot \pi$, then again $U(H) \cdot \rho$ is meager in $S(\pi)$ else it would have to be open and therefore intersect $U(H) \cdot \pi$. Thus we only need to verify that π is a turbulent point for the conjugacy action of $U(H)$ on $S(\pi)$. Keep in mind below that $S(\pi) \setminus U(H) \cdot \pi$ is dense in $S(\pi)$.

Claim. *For any non-empty open set W in $S(\pi)$ and any orthonormal set $e_1, \dots, e_p$ in H, there is an orthonormal set $e_1, \dots, e_p, e_{p+1}, \dots, e_q$ extending $e_1, \dots, e_p$, and $T \in U(H)$ such that*

(a) $T(e_i) \perp e_j, \forall i, j \leq q$,

(b) $T^2(e_i) = -e_i, \forall i \leq q$,

(c) $T = \mathrm{id}$ on $(H_0 \oplus T(H_0))^\perp$, where $H_0 = \langle e_1, \dots, e_q \rangle$,

(d) $T \cdot \pi \in W$.

Assume this and proceed to show that π is turbulent. Fix a basic nbhd of π in $S(\pi)$:

$$W = W_{\pi, \mathbf{e}, F, \epsilon} = \{\rho : \forall \gamma \in F \forall i \leq k | \langle \rho(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(e_i), e_j \rangle | < \epsilon\},$$

where $\mathbf{e} = e_1, \dots, e_k$ is orthonormal, $F \in \Gamma$ is finite symmetric, and $\epsilon > 0$. Let $e_1, \dots, e_k, e_{k+1}, \dots, e_p$ be an orthonormal basis for the span of

$$e_1, \dots, e_k, \pi(\gamma)(e_i), 1 \leq i \leq k, \gamma \in F,$$

and let $e_1, \dots, e_p, e_{p+1}, \dots, e_q, T$ be as in the claim (so that in particular $T \cdot \pi \in W$). It is clearly enough to find a continuous path $\{T_\theta\}_{0 \leq \theta \leq \pi/2}$ in $U(H)$ with $T_0 = 1, T_{\pi/2} = T$ and $T_\theta \cdot \pi \in W, \forall \theta$ (which clearly implies turbulence as in the proof of 5.1). Put

$$\begin{aligned} T_\theta(e_i) &= (\cos\theta) e_i + (\sin\theta) T(e_i), \\ T_\theta(T(e_i)) &= (-\sin\theta) e_i + (\cos\theta) T(e_i), \end{aligned}$$

for $i = 1, \dots, q$, and $T_\theta = \mathrm{id}$ on $(H_0 \oplus T(H_0))^\perp$, where $H_0 = \langle e_1, \dots, e_q \rangle$. Thus $T_0 = 1, T_{\pi/2} = T$ and we will verify that $T_\theta \cdot \pi \in W$.

Indeed for $i, j \leq k, \gamma \in F$,

$$\begin{aligned} \langle T_\theta \cdot \pi(\gamma)(e_i), e_j \rangle &= \langle \pi(\gamma)(T_\theta^{-1}(e_i)), T_\theta^{-1}(e_j) \rangle \\ &= \langle \pi(\gamma)((\cos\theta) e_i - (\sin\theta) T(e_i)), (\cos\theta) e_j - (\sin\theta) T(e_j) \rangle \\ &= (\cos^2\theta) \langle \pi(\gamma)(e_i), e_j \rangle + (\sin^2\theta) \langle \pi(\gamma)(T(e_i)), T(e_j) \rangle, \end{aligned}$$

since $\pi(\gamma)(T(e_i)) \perp e_j$ and $\pi(\gamma)(e_i) \perp T(e_j)$ (note that $\pi(\gamma)(T(e_i)) \perp e_j$ iff $T(e_i) \perp \pi(\gamma)^{-1}(e_j) = \pi(\gamma^{-1})(e_j)$, and $\gamma^{-1} \in F$). So if

$$A = \langle T_\theta \cdot \pi(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(e_i), e_j \rangle,$$

then

$$\begin{aligned} A &= -\sin^2\theta(\langle \pi(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(T(e_i)), T(e_j) \rangle) \\ &= -\sin^2\theta(\langle \pi(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(-T^{-1}(e_i)), -T^{-1}(e_j) \rangle) \\ &= -\sin^2\theta(\langle \pi(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(T^{-1}(e_i)), T^{-1}(e_j) \rangle) \\ &= -\sin^2\theta(\langle \pi(\gamma)(e_i), e_j) \rangle - \langle T\pi(\gamma)T^{-1}(e_i), e_j \rangle), \end{aligned}$$

therefore

$$|A| \leq |\langle T \cdot \pi(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(e_i), e_j \rangle| < \epsilon,$$

so $T_\theta \cdot \pi \in W$.

Proof of the Claim. Without loss of generality we can assume that W has the form

$$W \equiv W_{\sigma,\mathbf{e},F,\epsilon}$$

(where $W_{\sigma,\mathbf{e},F,\epsilon}$ is defined as above) for some $\mathbf{e} = e_1, \dots, e_p, e_{p+1}, \dots, e_q$, and some $\sigma \in S(\pi) \setminus U(H) \cdot \pi$ (as $S(\pi) \setminus U(H) \cdot \pi$ is dense in $S(\pi)$). Thus there is $T_n \in U(H)$ with $T_n \cdot \pi \to \sigma$ and $T_n \to^w 0$, thus also $T_n^{-1} \to^w 0$. Note that since $\langle T_n^{-1}(e_i), e_j\rangle \to 0, \forall i,j \leq q$, for all large enough n, $e_1, \dots, e_q, T_n^{-1}(-e_1), \dots, T_n^{-1}(-e_q)$ are linearly independent. Apply Gram-Schmid to $e_1, \dots, e_q, T_n^{-1}(-e_1), \dots, T_n^{-1}(-e_q)$ to get an orthonormal set $e_1, \dots, e_q, f_1^{(n)}, \dots, f_q^{(n)}$. Since $\langle T_n^{-1}(-e_i), e_j\rangle \to 0, \forall i,j \leq q$, it follows that $\|T_n^{-1}(-e_i) - f_i^{(n)}\| \to 0, \forall i \leq q$. Define $S_n \in U(H)$ by

$$S_n(e_i) = f_i^{(n)},$$
$$S_n(f_i^{(n)}) = -e_i,$$

$\forall i \leq q$, and

$$S_n = \text{id on } \langle e_1, \dots, e_q, f_1^{(n)}, \dots, f_q^{(n)}\rangle^{\perp}.$$

Then $S_n(e_i) = f_i^{(n)} \perp e_j, S_n^2(e_i) = S_n(f_i^{(n)}) = -e_i, \forall i,j \leq q$ and $\|S_n^{-1}(e_i) - T_n^{-1}(e_i)\| = \|T_n^{-1}(-e_i) - S_n^{-1}(-e_i)\| = \|T_n^{-1}(-e_i) - f_i^{(n)}\| \to 0, \forall i \leq q$.

Thus for all large enough $n, \|S_n^{-1}(e_i) - T_n^{-1}(e_i)\| < \epsilon/4, \forall i \leq q$, and $T_n \cdot \pi \in W_{\sigma,\mathbf{e},F,\epsilon/2}$, i.e.,

$$|\langle T_n\pi(\gamma)T_n^{-1}(e_i), e_j\rangle - \langle \sigma(\gamma)(e_i), e_j\rangle| < \epsilon/2,$$

$\forall i,j \leq q, \forall \gamma \in F$, or equivalently

$$|\langle \pi(\gamma)(T_n^{-1}(e_i)), T_n^{-1}(e_j)\rangle - \langle \sigma(\gamma)(e_i), e_j\rangle| < \epsilon/2.$$

Thus

$$|\langle \pi(\gamma)(S_n^{-1}(e_i)), S_n^{-1}(e_j)\rangle - \langle \sigma(\gamma)(e_i), e_j\rangle| < \epsilon,$$

$\forall i,j \leq q, \forall \gamma \in F$, i.e.,

$$S_n \cdot \pi \in W,$$

so $S_n = T$ works. □

It follows that for infinite-dimensional H the conjugacy action of $U(H)$ on $\text{Irr}(\Gamma, H)$ is generically turbulent iff there is $\pi \in \text{NLC}(\Gamma, H)$ with $S(\pi) = \text{Irr}(\Gamma, H)$, i.e., there is $\pi \in \text{NLC}(\Gamma, H)$ with dense conjugacy class in $\text{Irr}(\Gamma, H)$ or equivalently $\forall \rho \in \text{Irr}(\Gamma, H)(\rho \prec \pi)$.

There are countable groups Γ for which there is no dense conjugacy class in $\text{Irr}(\Gamma, H)$, e.g., products $\Gamma = G \times \Delta$, where Δ is non-trivial abelian and G is not abelian-by-finite.

Indeed for such a Γ and $\pi \in \text{Irr}(G, H)$, H infinite-dimensional, and $\chi \in \hat{\Delta}$ (i.e., χ a character on Δ), define $\pi_\chi \in \text{Irr}(\Gamma, H)$, by

$$\pi_\chi(g, \delta)(h) = \chi(\delta)\pi(g)(h).$$

Clearly the map

$$\Phi : \text{Irr}(G, H) \times \hat{\Delta} \to \text{Irr}(\Gamma, H)$$

is 1–1 and its range is conjugacy invariant, as $T \cdot \pi_\chi = (T \cdot \pi)_\chi$ for $\pi \in \mathrm{Irr}(G, H), T \in U(H)$. Moreover it is well known (since an abelian group is of type I) that every $\rho \in \mathrm{Irr}(\Gamma, H)$ is isomorphic to some π_χ (see Folland [Fo], 7.25), so Φ is a bijection and it is easy to check that it is actually a homeomorphism of $\mathrm{Irr}(G, H) \times \hat{\Delta}$ with $\mathrm{Irr}(\Gamma, H)$ which preserves conjugacy. So identifying π_χ with (π, χ), we can identify $\mathrm{Irr}(\Gamma, H)$ with $\mathrm{Irr}(G, H) \times \hat{\Delta}$ (on which $U(H)$ acts by conjugacy on the first coordinate). Clearly $S(\pi, \chi) = S(\pi) \times \{\chi\}$, so no conjugacy class is dense.

In the opposite direction, if $\mathrm{Irr}(\Gamma, H)$ is dense in $\mathrm{Rep}(\Gamma, H)$, then there is $\pi \in \mathrm{Irr}(\Gamma, H)$ with dense conjugacy class in $\mathrm{Rep}(\Gamma, H)$ and thus in $\mathrm{Irr}(\Gamma, H)$. (This is because both $\mathrm{Irr}(\Gamma, H)$ and

$$\{\pi \in \mathrm{Rep}(\Gamma, H) : \pi \text{ has a dense conjugacy class}\}$$

are dense G_δ in $\mathrm{Rep}(\Gamma, H)$.) So if moreover every conjugacy class in the space $\mathrm{Rep}(\Gamma, H)$ is meager, the conjugacy action of $U(H)$ in $\mathrm{Irr}(\Gamma, H)$ is generically turbulent. This is for example the case when $\Gamma = F_N, N \geq 2$, by H.7. (Note that $\mathrm{Rep}(F_N, H)$ can be identified with $U(H)^N$ and so the conjugacy classes in $\mathrm{Rep}(F_N, H)$ are meager, by the remark following 2.5.) The fact that when $\Gamma = F_N, N \geq 2$, there is $\pi \in \mathrm{Irr}(\Gamma, H)$ with dense conjugacy class in $\mathrm{Rep}(\Gamma, H)$, means that there is $\pi \in \mathrm{Irr}(\Gamma, H)$ so that $\rho \prec_Z \pi, \forall \rho \in \mathrm{Rep}(\Gamma, H)$, and in particular implies that there is an irreducible representation weakly containing any representation of Γ, a result first proved by Yoshizawa [Y].

More generally, call a countable group Γ *densely representable* (DR) if the set of all $\pi \in \mathrm{Rep}(\Gamma, H)$ with range $\pi(\Gamma) = \{\pi(\gamma) : \gamma \in \Gamma\} \subseteq U(H)$ dense in $U(H)$ is dense (and thus dense G_δ) in $\mathrm{Rep}(\Gamma, H)$. Since every $\pi \in \mathrm{Rep}(\Gamma, H)$ with dense range is clearly irreducible, it follows that if Γ is DR, then $\mathrm{Irr}(\Gamma, H)$ is dense G_δ in $\mathrm{Rep}(\Gamma, H)$. All free groups $F_N, N \geq 2$, are DR (see the second paragraph after 4.12). However, as we have seen earlier, no product group $G \times \Delta$, with Δ non-trivial abelian and G not abelian-by-finite, is DR. I do not know if, for example, $F_2 \times F_2$ is DR, in fact, apart from some small variations of free non-abelian groups, I do not know any other examples of DR groups.

In conclusion it is unclear what groups Γ admit a dense conjugacy class in $\mathrm{Irr}(\Gamma, H)$, what groups are DR and finally what groups Γ have the property that conjugacy classes in $\mathrm{Rep}(\Gamma, H)$ are meager.

Remark. Let Γ be a non-amenable group and denote by λ_Γ its left-regular representation on $\ell^2(\Gamma)$. Note that (in fact for *any* non-trivial Γ) λ_Γ is not irreducible; see de la Harpe [dlH], Appendix II. We consider the space $S(\lambda_\Gamma)$ of all irreducible representations (on an infinite-dimensional H) weakly contained in λ_Γ. Clearly this is a closed subspace of $\mathrm{Irr}(\Gamma, H)$. There is a dense conjugacy class in $S(\lambda_\Gamma)$ iff λ_Γ is weakly equivalent to an irreducible representation. It is known, see de la Harpe [dlH], 21. Proposition, that if Γ is ICC (i.e., Γ has infinite conjugacy classes), then λ_Γ is weakly equivalent to an irreducible representation and therefore $S(\lambda_\Gamma)$ has a dense conjugacy class.

If $\cong |S(\lambda_\Gamma)$ is not smooth, then $\mathrm{NLC}(\Gamma, H) \cap S(\lambda_\Gamma) \neq \emptyset$, and thus, by H.9, $S(\lambda_\Gamma)$ is not classifiable by countable structures.

Finally, there are many groups Γ for which the conjugacy action of $U(H)$ on $S(\lambda_\Gamma)$ is actually minimal (every orbit is dense). This means that *any* $\pi \in S(\lambda_\Gamma)$ is weakly equivalent to λ_Γ. Such groups are called C^**-simple* since this condition is equivalent to the simplicity of the reduced C^*-algebra of Γ. See de la Harpe [dlH] for an extensive discussion of these concepts. (See also the recent preprint Poznansky [Poz], where a characterization of the linear C^*-simple groups is obtained.) Among the C^*-simple groups are the free groups $F_N, N \geq 2$. Now if Γ is a C^*-simple group, every conjugacy class in $S(\lambda_\Gamma)$ is meager in $S(\lambda_\Gamma)$, otherwise it would have to be open in $S(\lambda_\Gamma)$, so $S(\lambda_\Gamma)$ would consist of a single conjugacy class, a contradiction. This means that the conjugacy action of $U(H)$ on $S(\lambda_\Gamma)$ is turbulent (and not just generically turbulent).

Remark. As in Section 15, we can also derive some consequences concerning connectedness. Let Γ be a countable group and H be infinite-dimensional. Let $\pi \in \mathrm{NLC}(\Gamma, H)$. Then if $C \subseteq S(\pi)$ contains the conjugacy class of π, C is path connected and locally path connected. Thus if there is $\pi \in \mathrm{NLC}(\Gamma, H)$ with dense conjugacy class in $\mathrm{Irr}(\Gamma, H)$, as in the case $\Gamma = F_N, N \geq 2$, then $\mathrm{Irr}(\Gamma, H)$ is path connected and locally path connected. On the other hand, if $\Gamma = G \times \Delta, \Delta$ a non-trivial abelian group with an element of finite order and G not abelian-by-finite, then $\hat{\Delta}$ is not connected (see Rudin [Ru2], 2.5.6), so $\mathrm{Irr}(\Gamma, H) \cong \mathrm{Irr}(G, H) \times \hat{\Delta}$ is not connected. I do now know what groups Γ have the property that $\mathrm{Irr}(\Gamma, H)$ is (path) connected.

(D) A representation $\pi \in \mathrm{Rep}(\Gamma, H)$, for infinite Γ, is called *ergodic* if it has no non-0 invariant vectors, *weak mixing* if it contains no non-0 finite dimensional subrepresentation, *mild mixing* if for every $h \in H, h \neq 0, \gamma_n \to \infty$, we have $\pi(\gamma_n)(h) \not\to h$, and *mixing* if it is a c_0*-representation*, i.e., $\lim_{\gamma\to\infty} \langle \pi(\gamma)(h), h\rangle = 0$, for every $h \in H$. Denote by $\mathrm{ERG}(\Gamma, H)$, $\mathrm{MMIX}(\Gamma, H)$, $\mathrm{WMIX}(\Gamma, H)$, $\mathrm{MIX}(\Gamma, H)$ the corresponding classes of representations. (We interpret these as being empty if Γ is finite.) Clearly

$$\mathrm{ERG}(\Gamma, H) \supseteq \mathrm{WMIX}(\Gamma, H) \supseteq \mathrm{MMIX}(\Gamma, H) \supseteq \mathrm{MIX}(\Gamma, H).$$

See Glasner [Gl2], Ch. 3 and Bergelson-Rosenblatt [BR] for other characterizations of these classes.

Using an argument similar to that of the (alternative) proof of 12.1, and Glasner [Gl2] or Bergelson-Rosenblatt [BR], we have the following fact.

Proposition H.11. *The sets* $\mathrm{ERG}(\Gamma, H)$ *and* $\mathrm{WMIX}(\Gamma, H)$ *are* G_δ *in the space* $\mathrm{Rep}(\Gamma, H)$.

Clearly $\mathrm{MIX}(\Gamma, H)$ is $\mathbf{\Pi}^0_3$ and $\mathrm{MMIX}(\Gamma, H)$ is $\mathbf{\Pi}^1_1$. However, by the paragraph before the Remark in Section 10, **(G)**, for infinite-dimensional H, $\mathrm{MMIX}(\Gamma, H)$ is not Borel, if Γ has an element of infinite order. Also $\mathrm{ERG}(\Gamma, H)$ has no interior in $\mathrm{Rep}(\Gamma, H)$, if H is infinite-dimensional.

(E) We now note some analogs of the characterizations in Sections 11, 12. Below 1_Γ is the trivial 1-dimensional representation of Γ. The equivalence of (a), (d), (e), (f), (g) in H.12 (i) below was also independently proved in Kerr-Pichot [KP].

Theorem H.12. *(i) The following are equivalent for any infinite countable group* Γ *and infinite-dimensional* H*:*

(a) Γ *has property* (T),

(b) $\mathrm{ERG}(\Gamma, H) = \{\pi \in \mathrm{Rep}(\Gamma, H) : 1_\Gamma \not\prec \pi\}$,

(c) $\mathrm{WMIX}(\Gamma, H) \subseteq \{\pi \in \mathrm{Rep}(\Gamma, H) : 1_\Gamma \not\prec \pi\}$,

(d) $\mathrm{ERG}(\Gamma, H)$ *is closed in* $\mathrm{Rep}(\Gamma, H)$,

(e) $\mathrm{ERG}(\Gamma, H)$ *is not dense in* $\mathrm{Rep}(\Gamma, H)$,

(f) $\mathrm{WMIX}(\Gamma, H)$ *is closed in* $\mathrm{Rep}(\Gamma, H)$,

(g) $\mathrm{WMIX}(\Gamma, H)$ *is not dense in* $\mathrm{Rep}(\Gamma, H)$.

(ii) The following are equivalent for any infinite countable group Γ *and infinite-dimensional* H*:*

(a) Γ *does not have* (HAP).

(b) $\mathrm{MIX}(\Gamma, H) \subseteq \{\pi \in \mathrm{Rep}(\Gamma, H) : 1_\Gamma \not\prec \pi\}$.

(iii) (**Bergelson-Rosenblatt [BR]**) *If an infinite countable group* Γ *has* (HAP), *then the set* $\mathrm{MIX}(\Gamma, H)$ *is dense in* $\mathrm{Rep}(\Gamma, H)$.

Proof. (i): (a) $\Leftrightarrow$ (b) follows from the definition of property (T). (b) $\Rightarrow$ (c) is clear and for $\neg(b) \Rightarrow \neg(c)$ see the remark following 11.2. So (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (d) follows exactly as in the proof of 12.2, (i) and (d) $\Rightarrow$ (e) $\Rightarrow$ (g) is clear. To prove that (d) $\Rightarrow$ (f), recall that for any $\pi_1 \in \mathrm{Rep}(\Gamma, H), \pi_2 \in \mathrm{Rep}(\Gamma, H_2)$, the tensor product $\pi_1 \otimes \pi_2 \in \mathrm{Rep}(\Gamma, H_1 \otimes H_2)$ is defined by

$$(\pi_1 \otimes \pi_2)(\gamma)(v_1 \otimes v_2) = \pi_1(\gamma)(v_1) \otimes \pi_2(\gamma)(v_2).$$

Then $\pi \in \mathrm{Rep}(\Gamma, H)$ is weak mixing iff for every $\rho \in \mathrm{Rep}(\Gamma, H')$ the tensor product $\pi \otimes \rho$ is ergodic (see Glasner [Gl2], 3.5). Since $(\pi, \rho) \mapsto \pi \otimes \rho$ is continuous, it is clear that (d) $\Rightarrow$ (f). Again (f) $\Rightarrow$ (g) is obvious.

Finally to prove (g) $\Rightarrow$ (a), assume that Γ does not have property (T) in order to show that $\overline{\mathrm{WMIX}(\Gamma, H)} = \mathrm{Rep}(\Gamma, H)$. Fix $\sigma \in \mathrm{Rep}(\Gamma, H)$. By (a) $\Leftrightarrow$ (c) there is $\pi_0 \in \mathrm{Rep}(\Gamma, H)$ with $1_\Gamma \prec \pi_0$ and $\pi_0 \in \mathrm{WMIX}(\Gamma, H)$. Then $\pi_0 \otimes \sigma$ is weak mixing.

Since $1_\Gamma \prec \pi_0$, we have $1_\Gamma \otimes \sigma \cong \sigma \prec \pi_0 \otimes \sigma$ (see Bekka-de la Harpe-Valette [BdlHV], F.3.2), so, by H.2, σ is in the closure of the set of isomorphic copies of $\infty \cdot (\pi_0 \otimes \sigma)$ in $\mathrm{Rep}(\Gamma, H)$. Since $\pi_0 \otimes \sigma$ is weak mixing, it is clear that $\infty \cdot (\pi_0 \otimes \sigma)$ is weak mixing, thus $\sigma \in \overline{\mathrm{WMIX}(\Gamma, H)}$, i.e., $\mathrm{Rep}(\Gamma, H) = \overline{\mathrm{WMIX}(\Gamma, H)}$.

(ii): (a) $\Leftrightarrow$ (b) is one of the equivalent definitions of (HAP) (see Cherix et al. [CCJJV]).

(iii): Assume now that Γ has (HAP). Then by (ii), (b) there is mixing $\pi \in \mathrm{Rep}(\Gamma, H)$ with $1_\Gamma \prec \pi$. Then for any $\sigma \in \mathrm{Rep}(\Gamma, H), \sigma \cong 1_\Gamma \otimes \sigma \prec \pi \otimes \sigma$ and $\pi \otimes \sigma$, $\infty \cdot (\pi \otimes \sigma)$ are mixing, thus $\sigma \in \overline{\mathrm{MIX}(\Gamma, H)}$. □

Corollary H.13. *If an infinite countable group* Γ *has property* (T) *and* H *is infinite-dimensional, then* $\mathrm{Irr}(\Gamma, H)$ *is nowhere dense.*

Proof. Clearly $\mathrm{Irr}(\Gamma, H) \subseteq \mathrm{ERG}(\Gamma, H)$, as H is infinite-dimensional. □

Remark. Bekka [B] has introduced the concept of an amenable unitary representation. One of its equivalent formulations is the following: $\pi \in \mathrm{Rep}(\Gamma, H)$ is amenable iff $1_\Gamma \prec \pi \otimes \overline{\pi}$. Here $\overline{\pi} \in \mathrm{Rep}(\Gamma, \overline{H})$ is the *conjugate representation* defined as follows: $\overline{H}$ is the conjugate space of H, which can be viewed as H with the same addition but with scalar multiplication and inner product defined by: $(\lambda, x) \mapsto \overline{\lambda}x, \langle x, y\rangle_{\overline{H}} = \langle y, x\rangle_H, \forall \lambda \in \mathbb{C}, \forall x, y \in H$. Finally $\overline{\pi} = \pi$ set theoretically.

Denote by $\mathrm{AMEN}(\Gamma, H)$ the set of amenable representations of Γ on H. Since $\pi \notin \mathrm{WMIX}(\Gamma, H)$ iff $1_\Gamma \leq \pi \otimes \overline{\pi}$ (see, e.g., Bekka-de la Harpe-Valette [BdlHV], A.1.11), it follows that if $\sim \mathrm{AMEN}(\Gamma, H) = \mathrm{Rep}(\Gamma, H) \setminus \mathrm{AMEN}(\Gamma, H)$, then $\sim \mathrm{AMEN}(\Gamma, H) \subseteq \mathrm{WMIX}(\Gamma, H)$. Bekka [B], 5.9 shows that if Γ has property (T), then $\sim \mathrm{AMEN}(\Gamma, H) = \mathrm{WMIX}(\Gamma, H)$ and Bekka-Valette [BV] show that conversely $\sim \mathrm{AMEN}(\Gamma, H) = \mathrm{WMIX}(\Gamma, H)$, implies that Γ has property (T). We note that this also follows immediately from H.12. Indeed if Γ does not have property (T), then, by H.12, (i), there is $\pi \in \mathrm{WMIX}(\Gamma, H)$ with $1_\Gamma \prec \pi$, so that π is amenable.

We have seen that Γ has property (T) iff $\mathrm{WMIX}(\Gamma, H)$ is closed. I do not know if it is true that Γ has property (T) iff $\sim \mathrm{AMEN}(\Gamma, H)$ is closed.

Remark. A countable group Γ is said to have *property* (FD) (see Lubotzky-Shalom [LS]) if the unitary representations which factor through a finite quotient of Γ are dense in $\mathrm{Rep}(\Gamma, H)$. It was shown in Lubotzky-Shalom [LS] that each free group F_n has property (FD). We note that this follows from the fact that $\mathrm{Aut}(X, \mu)$ is topologically locally finite together with Appendix E (the latter being also used in [LS]).

Observe that the property (FD) for F_n is equivalent to the following property of $U(H)$: The set of $(g_1, \dots, g_n) \in U(H)^n$ such that $\langle g_1, \dots, g_n\rangle$ is finite is dense in $U(H)^n$. To prove this, fix $g_1, \dots, g_n \in U(H)^n$, $e_1, \dots, e_m \in H$ and $\epsilon > 0$. We will find $(h_1, \dots, h_n) \in U(H)^n$ with $\langle h_1, \dots, h_n\rangle$ finite and $\|g_i(e_j) - h_i(e_j)\| < \epsilon, i = 1, \dots, n, j = 1, \dots, m$. By Appendix E, there is a standard Borel space (X, μ) such that H is a closed subspace of $L^2_0(X, \mu) = H_0$ and $g_1, \dots, g_n$ extend to $\bar{g}_1, \dots, \bar{g}_n \in \mathrm{Aut}(X, \mu) \subseteq U(L^2_0(X, \mu)) = U(H_0)$. Use now the topological local finiteness of $\mathrm{Aut}(X, \mu)$ to find $\tilde{g}_1, \dots, \tilde{g}_n \in U(H_0)$ with $\|g_i(e_j) - \tilde{g}_i(e_j)\| < \epsilon, \forall i \leq n, \forall j \leq m$ and $\langle \tilde{g}_1, \dots, \tilde{g}_n\rangle$ finite. We can find an isomorphism $\pi : H_0 \to H$ with $\pi(e_j) = e_j, \pi(g_i(e_j)) = g_i(e_j), \forall i \leq n, \forall j \leq m$. Finally let $h_i \in U(H)$ be the image of $\tilde{g}_i$ under π.

This clearly works as $h_i(e_j) = \pi(\tilde{g}_i(e_j)), g_i(e_j) = \pi(g_i(e_j))$ and so $\|h_i(e_j) - g_i(e_j)\| = \|\pi(\tilde{g}_i(e_j)) - \pi(g_i(e_j))\| = \|\tilde{g}_i(e_j) - g_i(e_j)\| < \epsilon$.

In connection with this argument, I do not know if $U(H)$ itself is topologically locally finite.

(F) Concerning the connection between $\mathrm{Rep}(\Gamma, H)$ and $A(\Gamma, X, \mu)$, let us call $\pi \in \mathrm{Rep}(\Gamma, H)$ (H infinite-dimensional) *realizable by an action* if there is $a \in A(\Gamma, X, \mu)$ such that $\pi \cong \kappa_0^a$ (= the Koopman representation on $L_0^2(X, \mu)$ associated with a).

Proposition H.14. *Let Γ be a countable group and assume that H is infinite-dimensional. Then the realizable by an action representations are dense in* $\mathrm{Rep}(\Gamma, H)$.

Proof. Let $\Gamma = \{\gamma_i\}_{i=1}^\infty$. Fix $\pi \in \mathrm{Rep}(\Gamma, H)$ and an orthonormal basis $\{e_j\}_{j=1}^\infty$ for H. Given $\epsilon > 0, n \geq 1, m \geq 1$, it is enough to find a realizable by an action $\rho \in \mathrm{Rep}(\Gamma, H)$ such that $\|\pi(\gamma_i)(e_j) - \rho(\gamma_i)(e_j)\| < \epsilon, \forall i \leq n, j \leq m$. By E.1 we can assume that H is a closed subspace of $L_0^2(X, \mu)$ and there is $a \in A(\Gamma, X, \mu)$ with κ_0^a extending π. Let H_1 be a finite-dimensional subspace of H which contains all $e_j, \pi(\gamma_i)(e_j), i \leq n, j \leq m$. Then find an isomorphism $T : L_0^2(X, \mu) \to H$ such that $T|H_1 =$ identity. Let $\rho \in \mathrm{Rep}(\Gamma, H)$ be the image of κ_0^a under T, so that clearly ρ is realizable by an action. We have for $i \leq n, j \leq m$,

$$\begin{aligned} \rho(\gamma_i)(e_j) &= T(\kappa_0^a(\gamma_i)(e_j)) \\ &= T(\pi(\gamma_i)(e_j)) \\ &= \pi(\gamma_i)(e_j), \end{aligned}$$

so we are done. □

Corollary H.15. *Let Γ be a countable group and H be infinite-dimensional. Let $P \subseteq \mathrm{Rep}(\Gamma, H)$ be invariant under conjugacy and assume that $\{a \in A(\Gamma, X, \mu) : \exists \pi \in P(\pi \cong \kappa_0^a)\}$ is dense in $(A(\Gamma, X, \mu), w)$. Then P is dense in* $\mathrm{Rep}(\Gamma, H)$.

Proof. We can assume that $H = L_0^2(X, \mu)$, in which case $(A(\Gamma, X, \mu), w)$ is a closed subspace of $\mathrm{Rep}(\Gamma, H)$ (via the identification of $a \in A(\Gamma, X, \mu)$ with κ_0^a). Then H.14 shows that the saturation of $A(\Gamma, X, \mu)$ under the conjugacy action of $U(H)$ is dense in $\mathrm{Rep}(\Gamma, H)$. Our hypothesis implies that $P \cap A(\Gamma, X, \mu)$ is dense in $(A(\Gamma, X, \mu), w)$ therefore, as the conjugacy action is continuous, P is dense in $\mathrm{Rep}(\Gamma, H)$. □

Since the set of realizable by an action $\pi \in \mathrm{Rep}(\Gamma, H)$ is invariant under conjugacy, it is either meager or comeager in $\mathrm{Rep}(\Gamma, H)$. When Γ is torsion-free abelian, it is not hard to see that it is meager. Indeed if π is realizable by an action, then the maximal spectral type σ of π (see Appendix F) is symmetric up to measure equivalence, $\sigma \sim \sigma^{-1}$, where $\sigma^{-1}(A) = \sigma(A^{-1})$, for A a Borel subset of $\hat{\Gamma}$ (see, e.g., Nadkarni [Na], p. 21 or Lemańczyk [Le], pp. 83–84, for the $\Gamma = \mathbb{Z}$ case). On the other hand, the set of $\pi \in$

$\mathrm{Rep}(\Gamma, H)$ whose maximal spectral type σ satisfies $\sigma \perp \sigma^{-1}$ is G_δ, since there is continuous $\pi \mapsto \sigma_\pi$ with σ_π the maximal spectral type of π (see Appendix F, **(B)**), and $\perp$ on $P(\hat{\Gamma})$ is a G_δ relation (note that $\mu \perp \nu$ iff $\mu \wedge \nu = 0$ iff $\inf\{\int f d\mu + \int (1-f) d\nu : f \in C(\hat{\Gamma}), 0 \leq f \leq 1\} = 0$). Now one can also see that the set of all $\pi \in \mathrm{Rep}(\Gamma, H)$ whose maximal spectral type σ satisfies $\sigma \perp \sigma^{-1}$ is dense in $\mathrm{Rep}(\Gamma, H)$, which completes the proof. To verify this, for each $\mu \in P(\hat{\Gamma})$ consider the representation π_μ of Γ on $L^2(\hat{\Gamma}, \mu)$ defined by $\pi_\mu(\gamma)(f) = \bar{\gamma} f$, where $\gamma \in \Gamma$ is viewed as a character of $\hat{\Gamma}$ (see Appendix F). Clearly μ is the maximal spectral type of π_μ. Now any $\pi \in \mathrm{Rep}(\Gamma, H)$ is isomorphic to a direct sum $\bigoplus_n \pi_{\mu_n}$ (see Appendix F), and by approximating probability measures on $\hat{\Gamma}$ by probability measures with finite support, we see that for each $\mu_1, \mu_2, \dots \in P(\hat{\Gamma})$, there is $\nu_1, \nu_2, \dots, \in P(\hat{\Gamma})$ such that each ν_n has finite support, $\pi_{\mu_n} \prec \pi_{\nu_n}$, and $\sum 2^{-n} \nu_n \perp (\sum 2^{-n} \nu_n)^{-1}$. Thus $\pi \cong \bigoplus_n \pi_{\mu_n} \prec \bigoplus_n \pi_{\nu_n}$. Moreover the maximal spectral type of $\bigoplus_n \pi_{\nu_n}$ and thus of $\infty \cdot (\bigoplus_n \pi_{\nu_n})$ is $\sigma = \sum_n 2^{-n} \nu_n$ and, as $\sigma \perp \sigma^{-1}$, it follows from H.2 that π is in the closure of the set of all $\rho \in \mathrm{Rep}(\Gamma, H)$ whose maximal spectral type σ satisfies $\sigma \perp \sigma^{-1}$.

In the case $\Gamma = \mathbb{Z}$, one can also give a proof along the lines of the proof of 2.5 and the remark following it. We take $H = L^2_0(X, \mu)$ and we identify $\mathrm{Rep}(\mathbb{Z}, H)$ with $U(L^2_0(X, \mu))$. Then the set of representable by an action $U \in U(L^2_0(X, \mu))$ is the saturation under conjugacy of $\mathrm{Aut}(X, \mu)$, which we view as a closed subgroup of $U(L^2_0(X, \mu))$. Assuming this set is comeager, towards a contradiction, we can find a Borel function $U \mapsto f_U$ on $U(H)$ such that $\forall^* U (f_U U f_U^{-1} \in \mathrm{Aut}(X, \mu))$. Fix now a real-valued $\varphi \in H$ with $\|\varphi\| = 1$ and $\{\xi_n\} \subseteq H$ dense. Then there is some n and open nonempty $W \subseteq U(H)$ such that, letting $\xi = \xi_n$, we have

$$\forall^* U \in W \left(\|f_U^{-1}(\varphi) - \xi\| < \frac{1}{8} \right),$$

so that, in particular, $\frac{7}{8} < \|\xi\| < \frac{9}{8}$. Let now

$$\Omega = \left\{ U \in U(H) : \exists n \left[\mathrm{Im} \, \langle U^n(\xi), \xi \rangle > \frac{9}{16} \right] \right\}.$$

This is clearly open and we claim that is also dense. Using the notation of the Remark following 2.5, it is enough to show that it is dense in each $U(m)$ and this follows as in the argument in that remark by approximating elements of $U(m)$ by nth roots of the operator iI.

It follows that there is $U \in U(H)$ such that $\|f_U^{-1}(\varphi) - \xi\| < \frac{1}{8}, V = f_U U f_U^{-1} \in \mathrm{Aut}(X, \mu)$ and $\mathrm{Im} \, \langle U^n(\xi), \xi \rangle > \frac{9}{16}$, for some n. Thus we have

$$\mathrm{Im} \, \langle V^n(f_U(\xi)), f_U(\xi) \rangle > \tfrac{9}{16}$$

and, since $\|f_U(\xi) - \varphi\| < \frac{1}{8}$, it follows that

$$|\langle V^n(f_U(\xi)), f_U(\xi) \rangle - \langle V^n(\varphi), \varphi \rangle| < \tfrac{1}{8} \cdot \|\xi\| + \tfrac{1}{8} \cdot \|\varphi\| < \tfrac{1}{8} \cdot \tfrac{9}{8} + \tfrac{1}{8} < \tfrac{1}{3},$$

therefore $\operatorname{Im} \langle V^n(\varphi), \varphi \rangle > \frac{9}{16} - \frac{1}{3} > 0$, contradicting the fact that the number $\langle V^n(\varphi), \varphi \rangle$ is real, as φ is real and $V \in \operatorname{Aut}(X, \mu)$.

Problem H.16. *Let Γ be an infinite countable group and H be infinite-dimensional. Is the set of realizable by an action $\pi \in \operatorname{Rep}(\Gamma, H)$ meager in $\operatorname{Rep}(\Gamma, H)$?*

APPENDIX I

Semidirect products of groups

(A) Let A, G be groups and let A act on G by automorphisms. Denote the action by $a \cdot g$. Then the *semidirect product* $A \ltimes G$ is the group defined as follows: The space of $A \ltimes G$ is the product $A \times G$. Multiplication is defined by

$$(a_1, g_1)(a_2, g_2) = (a_1 a_2, g_1(a_1 \cdot g_2)) \tag{1}$$

There is a canonical action of A by automorphisms on $A \ltimes G$ that extends the action of A on G, namely

$$a \cdot (b, g) = (aba^{-1}, a \cdot g)$$

An *affine automorphism* of G is any map ψ of the form

$$\psi(g) = h\varphi(g),$$

where $h \in G$ and φ is an automorphism of G. Clearly any such ψ can be identified with the pair $(\varphi, h), \varphi$ being the *automorphic part* of ψ and $h = \psi(1)$ the *translation part*. Note that if $\psi_1 = (\varphi_1, h_1), \psi_2 = (\varphi_2, h_2)$, then

$$\begin{aligned} \psi_1(\psi_2(g)) &= h_1\varphi_1(\psi_2(g)) \\ &= h_1\varphi_1(h_2\varphi_2(g)) \\ &= h_1\varphi_1(h_2)\varphi_1\varphi_2(g), \end{aligned}$$

i.e., $\psi_1\psi_2 = (\varphi_1\varphi_2, h_1\varphi_1(h_2))$, thus the group of affine automorphisms of G can be identified with the semidirect product of the automorphism group of G with G, where the automorphism group of G acts by evaluation on G.

If now A acts by automorphisms on G, then $A \ltimes G$ acts by affine automorphisms on G via

$$(a, g) \cdot h = g(a \cdot h),$$

and if A acts on G faithfully (i.e., $a \cdot g = g, \forall g \Rightarrow a = 1$), so that A can be viewed as a group of automorphisms of G, then $A \ltimes G$ acts faithfully on G, so $A \ltimes G$ can be viewed as a group of affine automorphisms of G.

Example. Let H be a separable real Hilbert space, viewed as an additive group, $A = O(H)$ its orthogonal group and consider the action of $O(H)$ on H by evaluation. Every isometry φ of H can be uniquely written as $\varphi = T + b$, where $T \in O(H)$ and $b = \varphi(0) \in H$. Thus identifying φ with the pair (T, b), we see that the isometry group of H is identified with $O(H) \ltimes H$.

Example. Let $G = (\mathrm{MALG}_\mu, \Delta)$ and consider the metric $d_\mu(A, B) = \mu(A\Delta B)$ on G. Then $A = \mathrm{Aut}(X, \mu)$ acts by isometric automorphisms on G. Every isometry of $(\mathrm{MALG}_\mu, d_\mu)$ can be uniquely written as $\varphi = T\Delta B$, where $T \in \mathrm{Aut}(X, \mu), B = \varphi(\emptyset) \in \mathrm{MALG}_\mu$ (see Section 1, **(B)**). Thus the isometry group of $(\mathrm{MALG}_\mu, d_\mu)$ is identified with $A \ltimes G$.

Denote now by $\mathrm{Hom}(\Gamma, K)$ the set of homomorphisms of a group Γ into a group K. Any $\psi \in \mathrm{Hom}(\Gamma, A \ltimes G)$ can be viewed as a pair $\psi = (\varphi, \alpha)$, where $\psi(\gamma) = (\varphi(\gamma), \alpha(\gamma))$. Then $\varphi \in \mathrm{Hom}(\Gamma, A)$ and $\alpha : \Gamma \to G$ satisfies

$$\alpha(\gamma\delta) = \alpha(\gamma)(\varphi(\gamma) \cdot \alpha(\delta)).$$

Such an α is called a *cocycle* associated to the homomorphism φ. We denote by $Z^1(\varphi)$ the set of these cocycles. Thus

$$\mathrm{Hom}(\Gamma, A \ltimes G) = \{(\varphi, \alpha) : \varphi \in \mathrm{Hom}(\Gamma, A) \ \& \ \alpha \in Z^1(\varphi)\}.$$

For each fixed $\varphi \in \mathrm{Hom}(\Gamma, A)$, $Z^1(\varphi)$ gives all possible "extensions" of φ to a homomorphism into $A \ltimes G$. We can view $\varphi \in \mathrm{Hom}(\Gamma, A)$ as a representation of Γ as a group of automorphisms of G and then $Z^1(\varphi)$ gives representations of Γ as a group of affine automorphisms of G with automorphic part equal to φ. If for instance $A = \mathrm{Aut}(G)$ is the automorphism group of G acting by evaluation on G, then $Z^1(\varphi)$ gives all possible representations of Γ as a group of affine automorphisms of G with automorphic part equal to φ.

A *coboundary* of $\varphi \in \mathrm{Hom}(\Gamma, A)$ is a cocycle of the form

$$\alpha(\gamma) = g^{-1}(\varphi(\gamma) \cdot g),$$

for some $g \in G$. Their set is denoted by $B^1(\varphi)$. The group G acts on $Z^1(\varphi)$ by

$$(g \cdot \alpha)(\gamma) = g\alpha(\gamma)(\varphi(\gamma) \cdot g^{-1})$$

Two cocycles $\alpha, \beta \in Z^1(\varphi)$ are called *cohomologous*, in symbols

$$\alpha \sim \beta,$$

if $\exists g \in G(g \cdot \alpha = \beta)$. Thus the coboundaries are the cocycles cohomologous to the trivial cocycle 1. The quotient space

$$H^1(\varphi) = Z^1(\varphi)/\sim$$

is called the *(1st)-cohomology space* of φ. When G is abelian, $Z^1(\varphi)$ is an abelian group under pointwise multiplication, $B^1(\varphi)$ is a subgroup and $H^1(\varphi) = Z^1(\varphi)/B^1(\varphi)$ is the *(1st)-cohomology group* of φ.

Note here the following simple fact.

Proposition I.1. *$\alpha \in Z^1(\varphi)$ is a coboundary iff the affine representation of Γ in G, $\psi = (\varphi, \alpha)$, associated to φ, α, has a fixed point.*

Proof. If ψ has a fixed point $g^{-1} \in G$, then for every $\gamma, \psi(\gamma)(g^{-1}) = g^{-1}$, so $\alpha(\gamma)(\varphi(\gamma) \cdot g^{-1}) = g^{-1}$, thus

$$\begin{aligned}\alpha(\gamma) &= g^{-1}(\varphi(\gamma) \cdot g^{-1})^{-1} \\ &= g^{-1}(\varphi(\gamma) \cdot g),\end{aligned}$$

so $\alpha \in B^1(\varphi)$. Conversely, if $\alpha \in B^1(\varphi)$, then $\alpha(\gamma) = g^{-1}(\varphi(\gamma) \cdot g)$ for some g and, reversing the above steps, g^{-1} is a fixed point of ψ. □

Example. Let H be a separable real Hilbert space. Then $O(H) \ltimes H$ is its isometry group. A homomorphism π of Γ into $O(H)$ is an orthogonal representation of Γ in H. Then $Z^1(\pi)$ consists of all the cocycles $b : \Gamma \to H$ (i.e., maps satisfying $b(\gamma\delta) = b(\gamma) + \pi(\gamma)(b(\delta))$) and the pairs (π, b) with $b \in Z^1(\pi)$ give all affine isometric representations of Γ in H with orthogonal part π. Coboundaries are the cocycles of the form $b(\gamma) = \pi(\gamma)(h) - h$ for some $h \in H$. Clearly $Z^1(\pi)$ is an abelian group (under pointwise addition), in fact a real vector space, and $B^1(\pi)$ a subspace. Then $H^1(\pi) = Z^1(\pi)/B^1(\pi)$ is the (1st-)cohomology group (or vector space) of the representation π. It vanishes (i.e., $H^1(\pi) = \{0\}$) iff all isometric extensions of π admit fixed points. For more on the cohomology of representations, especially in connection with property (T), see Bekka-de la Harpe-Valette [BdlHV].

Remark. When A, G are Polish groups and A acts continuously on G, then the semidirect product $A \ltimes G$ with the product topology is clearly a Polish group.

(B) Sometimes one uses a different rule of multiplication for the semidirect product $A \ltimes G$, namely

$$(a_1, g_1)(a_2, g_2) = (a_1 a_2, (a_2^{-1} \cdot g_1) g_2). \tag{2}$$

It is easy to see however that

$$(a, g) \mapsto (a, a^{-1} \cdot g)$$

is an isomorphism between $A \ltimes G$ using the multiplication (1) with $A \ltimes G$ using the multiplication (2). It is more convenient to use the second form in the context of the theory of cocycles for group actions.

When we use (2) as the multiplication in $A \ltimes G$, it is easy to see that the canonical action of A on $A \ltimes G$ is still defined by: $a \cdot (b, g) = (aba^{-1}, a \cdot g)$. The cocycles associated to a homomorphism $\varphi \in \mathrm{Hom}(\Gamma, A)$ are now the maps $\alpha : \Gamma \to G$ that satisfy

$$\alpha(\gamma\delta) = (\varphi(\delta)^{-1} \cdot \alpha(\gamma))\alpha(\delta)$$

and the coboundaries are the cocycles of the form

$$\alpha(\gamma) = (\varphi(\gamma)^{-1} \cdot g)g^{-1}.$$

Finally G acts on $Z^1(\varphi)$ by

$$(g \cdot \alpha)(\gamma) = (\varphi(\gamma)^{-1} \cdot g)\alpha(\gamma)g^{-1}.$$

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